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ABSTRACT

Reported are detailed recommendations on the mathematics offerings in two year colleges which include the usual range of university parallel programs. The basic program is considered under four headings: Calculus Preparatory, Calculus and Linear Algebra, Business and Social Science, and Teacher Training. Twelve courses are discussed and outlined in some detail, seven of them in the basic categories and the remainder optional additional offerings intended for accelerated students. Problems of implementation are also considered. (MM)

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M A T H E M A T I C A L   A S S O C I A T I O N   O F   A M E R I C A

A TRANSFER CURRICULUM IN MATHEMATICS FOR TWO YEAR COLLEGES

COMMITTEE ON THE UNDERGRADUATE
PROGRAM IN MATHEMATICS

1969

A TRANSFER CURRICULUM IN MATHEMATICS

FOR TWO YEAR COLLEGES

The Committee on the Undergraduate Program in Mathematics is a committee of the Mathematical Association of America charged with making recommendations for the improvement of college and university mathematics curricula at all levels and in all educational areas. Financial support for CUPM has been provided by the National Science Foundation.

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CHAPTER I
BACKGROUND

§1. Two Year Colleges*.

It is impossible not to be impressed, perhaps a little overwhelmed, by the growth and diversity of the two-year college sector of American higher education. To cite only a few facts indicating the rate of growth: new institutions are being added at the rate of one each week to the approximately 900 that now exist; the largest student body in any Florida educational institution is that of Miami-Dade Junior College; 86 percent of all freshmen entering California public colleges in 1966 enrolled in two-year colleges; Seattle Community College opened in 1967 with an initial enrollment of 12,688 students. The extraordinary rate of growth in number of institutions and size of individual institutions is matched by their diversity, of size, purpose, make-up of student body, type of institutional control, variety of curricula, etc. A glance at a random selection of two-year college catalogs reveals this diversity quite strikingly. Further aspects of it can be noted in the annual directories of the American Association of Junior Colleges, and others are described and documented in a 1967 report to Congress from the National Science Foundation.**

Anyone interested in mathematical education who is accustomed to the comparatively stable national scene in either the elementary and secondary schools or in the four-year colleges and universities may find the two-year college picture bewildering. There is an immense variety of programs, including not only those comparable to the first two years in a university, but also a large assortment of technical, occupational and semi-professional programs as well as programs for general education and remedial study. The programs offered vary widely from school to school; the two-year college is often much more responsive to the community it serves and somewhat less responsive to tradition than the other institutions.

The student body attracted by these programs or by the convenience and economy of two-year college education exhibits extremes of age and maturity, background and preparation, and ability and motivation. More than two-thirds of these students regard themselves

*The name "two-year college" is intended to tie in with the "lower division" at a college or university. Students at any of these schools may spend more or less than two years on their "lower division" work.

**The Junior College and Education in the Sciences, U. S. Government Printing Office, Washington, 1967.

initially as transfer students,[1] but only about one-third eventually proceed to a four-year college.[2] This phenomenon is reflected in the figures on two-year college course enrollments in mathematical science compiled by the Conference Board of the Mathematical Sciences:[3]

"Of these course enrollments 324,000 were classified by the institutions themselves as being in freshman courses and only 24,000 as being in sophomore courses. This is a much sharper drop than in four-year colleges."

This survey and the NSF report contain revealing data on the faculty. They describe a faculty which includes many part-time teachers, has a broad range of academic preparation, is recruited from a variety of sources (graduate schools 24%, colleges and universities 17%, high schools 30%, other sources 29%)[4] and is highly mobile ("About 25 percent of all junior college professors were new to their particular campuses in 1964-65.").[5]

The situation sketched here has attracted the attention of many organizations concerned with the improvement of American higher education, of which CUPM is but one.

§2. CUPM.

The Mathematical Association of America (MAA) is the national professional organization concerned with the teaching of mathematics on the college level. The Committee on the Undergraduate Program in Mathematics (CUPM) is one of the standing committees of the MAA. Simultaneously, CUPM is one of the eight college commissions in the sciences, supported by the National Science Foundation, "to serve as instruments through which leading scientists can provide stimulation, guidance, and direction to the academic community in the improvement of undergraduate instruction."

The early curriculum recommendations published by CUPM dealt with specific aspects of education in mathematics, such as the training of physical science and engineering students, the training of teachers of elementary and high school mathematics and the undergraduate preparation of graduate students of mathematics.

1. NSF Report, p. 92.

2. NSF Report, p. 5.

3. Report of the Survey Committee. Volume I of Aspects of Undergraduate Training in the Mathematical Sciences. Available at $1.75 from CBMS, 2100 Pennsylvania Avenue, N.W., Suite 834, Washington, D. C. 20037.

4. NEA figures for 1963-64 and 1964-65 in all fields (see CBMS survey, p. 76).

5. NSF Report, p. 71.

It eventually became clear that CUPM had a responsibility to show how an overall curriculum could be constructed which was within the capabilities of a fairly small college and allowed for the implementation of its various special programs. A study of this problem led to the 1965 report A GENERAL CURRICULUM IN MATHEMATICS FOR COLLEGES (GCMC).* This report has received wide publicity through CUPM regional conferences and Section meetings of the MAA. Its major features have met with general approval and are having a growing influence on college textbook and curriculum reforms in tangible as well as in many intangible ways. <u>Regarding this influence, it must be stressed that CUPM does not write curricula in order to prescribe what courses a department should teach, but rather to offer generalized models for discussion and to provide the framework for meaningful dialogue</u> (within schools, between schools and on a broader scale) <u>on serious curricular problems. These models are meant to be both realistic and forward looking.</u>

The present report is a natural extension of these efforts. In addition it provides, together with GCMC, an aid to articulation of two-and four-year programs.

§3. The Present Report.

This initial CUPM report on two-year colleges is aimed at the transfer programs only. Some history of the study and reasons for this choice are outlined here.

The CUPM Panel on Mathematics in Two Year Colleges was formed in 1966 following some preliminary study of the need and potential in this area for the kind of activities which CUPM had successfully pursued in other areas: curriculum studies and recommendations, regional conferences discussing local problems within the framework of the recommendations, and individual visits to institutions by CUPM consultants.**

The Panel, consisting of mathematicians with extensive experience in various phases of education, at four-year colleges and universities as well as at two-year colleges, sponsored a series of

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*The report presents a curriculum which can be taught by a staff of four or five and which includes courses for the various special programs. (An exception is the special course sequence for elementary teacher training; staff for these courses is not included in the estimate.)

**The CUPM Consultants Bureau arranges visits by consultants to individual institutions in response to invitations from the latter. A brochure describing the service and giving biographies of current consultants is distributed annually from the CUPM Central Office, P. O. Box 1024, Berkeley, California 94701.

meetings. Representatives of a wide spectrum of two-year colleges were called upon and provided much detailed information to supplement the Panel's studies of the national scene, with its manifold local variations. These meetings were supplemented by participation in other activities related to the problem, such as meetings of the NSF Intercommission Panel on Two Year Colleges, meetings of various organizations of two-year college mathematics teachers, individual visits to institutions, and a wealth of personal contacts. The Panel also sought the advice of experts in various areas serving on other CUPM Panels.

The Panel was divided into subpanels during this study phase, to concentrate on three topics: mathematics for general education, mathematics for technical-occupational programs, and mathematics for four-year college transfer programs (in all disciplines).

Many two-year college teachers who consulted with the Panel expressed the opinion that guidance was most needed on the first two topics. However, it became increasingly clear as the study progressed, first, that considerable overlap existed in the problems in these three areas, and second, that an initial concentration on the third topic was most natural, both logically and from the viewpoint of CUPM's customary methods of operation:

Transfer programs are offered at almost all two-year colleges, and they determine the basic mathematics offerings. Local variations are least in this area.

A workable, imaginative solution to the problem of a transfer curriculum would provide the natural first step and could go far towards solution of the problems in the other two areas. It would answer the needs of the great majority of the students (two-thirds intend to transfer) especially if it kept open a variety of options for these students, at least through the first year.

The transfer curriculum lends itself more readily to a curriculum study with follow-up conferences. The other two problems involve additional considerations, such as pedagogical techniques and the special needs of students with very specific goals, which make this general approach less effective. CUPM's wide experience with the GCMC report and conferences (frequently involving two-year college teachers) had clearly indicated a need and a demand for similar efforts suitably adapted to the realities of two-year colleges.

The present report differs from GCMC in several major respects. Among these is the fact that it includes more detailed course descriptions, discussions of the rationale for choices that were made, and frequent comments on how topics might be taught. It also includes comments on implementation; the use and influence of the computer, articulation, etc. (see Chapter 5). Finally, it includes explicit recommendations on teacher training (see Chapter 3).

§4. Staff.

The CBMS Survey mentioned earlier reports the following data on staff. The 167 two-year colleges having enrollments of more than 2,000 employed 44 full-time mathematics staff members with a doctorate in some field (not necessarily a mathematical science); 451 with a master's degree plus one additional year of graduate study in some field; and 439 with a master's degree in mathematics. The corresponding figures for the 543 schools of under 2,000 enrollment are 66, 306 and 513 respectively. There are, of course, many additional staff members with less academic preparation or who teach part time.

From these figures we can compute roughly the average number of staff members with academic preparation in the above range: five or six for the larger schools and one or two for the smaller schools. The corresponding figures when part-time faculty are included are eight and two, roughly.

The present report takes account of these facts by outlining a group of basic university parallel courses that provides the necessary offerings for normal transfer programs (including elementary teacher training). It was prepared with the small department in mind. A larger department would have the potential for offering some additional courses to supplement these. Several possibilities for such courses are suggested in the report.

The important thing is that we believe our basic courses can be taught by approximately the equivalent of two full-time staff members. In Chapter 5, §1, we give an illustration of how this can be done. The actual number of teachers required in a particular case will depend, of course, on class size, teaching load, and other such factors.

It is perhaps the most striking common feature of two-year colleges that their faculties are highly student-oriented, much more so than the more discipline-oriented faculties at the four-year colleges and universities.[1] Nevertheless, it is important not to overlook problems of academic qualifications. In particular, a report such as this or the GCMC report would exist in a partial vacuum if there were no accompanying considered statements concerning the academic qualifications desired of the teachers of the programs. Such a report has been written to accompany the GCMC[2] and another report outlining a graduate program to achieve the proposed training has been prepared.[3]

1. See, for example, R. H. Garrison; Junior College Faculty: Issues and Answers, American Association of Junior Colleges, 1967.

2. Qualifications for a College Faculty in Mathematics (1967).

3. A Beginning Graduate Program in Mathematics (1969).

Simultaneously with its decision to begin with the university parallel study, CUPM organized an ad hoc Committee on the Qualifications of a Two Year College Faculty. The report of this committee* will be a companion to this report.

§5. The Proposed Programs.

The courses which CUPM proposes are described in detail in Chapters 2, 3, and 4. We summarize them here under the broad classification of Basic Offerings and Additional Offerings, and discuss some of the programs which such offerings permit.

Where appropriate, we draw attention to the comparable course in the GCMC report. The latter, of course, serves as an alternative to be considered by those two-year colleges that are structured on purely university parallel lines.

BASIC OFFERINGS

I. Calculus Preparatory

(a) Elementary Functions and Coordinate Geometry, Mathematics 0 (as in GCMC).

(b) Elementary Functions and Coordinate Geometry, with Algebra and Trigonometry, Mathematics A.

One or both of these should be offered by every two-year college.

II. Calculus and Linear Algebra

(a) Introductory Calculus, Mathematics B. (An intuitive course covering the basic concepts of single variable calculus. Similar to GCMC Mathematics 1.)

(b) Mathematical Analysis, Mathematics C. (A more rigorous course completing the standard calculus topics, as in GCMC Mathematics 2, 4.)

(c) Linear Algebra, Mathematics L. (An elementary treatment similar to GCMC Mathematics 3, but parallel to, rather than preceding, the last analysis course.)

Categories I and II constitute the basic pre-science offerings and should be offered by every two-year college with a transfer program.

*Qualifications for a Two Year College Faculty in Mathematics (1969).

III. Business and Social Science

Probability and Statistics, Mathematics PS. (An introductory course stressing basic statistical concepts.)

IV. Teacher Training

Structure of the Number System, Mathematics NS. (A year course as recommended by the Panel on Teacher Training for the preparation of elementary school (Level I) teachers. The second year of preparation, algebra and geometry, should also be offered wherever possible (see Chapter 3).)

This completes the minimal set of offerings envisioned for a two-year college that embraces the usual range of university parallel programs.

The courses A, C and NS are full year courses. If we consider a semester system and use the obvious notation, the basic program is represented by the following diagram.

BASIC OFFERINGS

Calculus Preparatory → Calculus-Linear Algebra

$$\boxed{0 \text{ or } A_1 \longrightarrow A_2} \longrightarrow \boxed{B \longrightarrow C_1 \longrightarrow \left\{\begin{matrix} C_2 \\ L \end{matrix}\right.}$$

Business and Social Science

$$\boxed{PS}$$

Elementary Teacher Training

$$\boxed{NS_1 \longrightarrow NS_2 \longrightarrow \left\{\begin{matrix} \text{Algebra} \\ \text{Geometry} \end{matrix}\right.}$$

ADDITIONAL OFFERINGS (Optional)

1. Finite Mathematics, Mathematics FM. (A course of considerable interest and utility, especially for students of the non-physical sciences.)

2. Intermediate Differential Equations, Mathematics DE.

3. Differential Equations and Advanced Calculus, Mathematics DA.

4. Probability Theory, Mathematics PR. (A calculus based course as in GCMC, Mathematics 2P.)

5. Numerical Analysis, Mathematics NA.

Selections from this list are suggested to round out a department's offerings according to its special needs and interests.

PROGRAMS*

We see this combination of courses as providing a variety of tracks to meet the needs of students with different educational goals and mathematical backgrounds and abilities. Some of the possibilities are described here.

(i) For the student in physical science, engineering and mathematics (including future secondary school teachers) the program should include B, C, and L. A sequence for well prepared students is provided by the semester courses

$$B \longrightarrow C_1 \longrightarrow C_2 \longrightarrow L,$$

plus possible additional courses. A slightly less well prepared student can achieve this by taking the sequence of courses

$$0 \longrightarrow B \longrightarrow C_1 \longrightarrow (C_2 \text{ and } L).$$

The student who is not prepared for Mathematics 0 at the outset requires an extra semester or summer session to complete calculus at the two-year college, as indicated by the program

$$A_1 \longrightarrow A_2 \longrightarrow B \longrightarrow C_1 \longrightarrow (C_2 \text{ and } L).$$

(ii) Students of the biological, management and social sciences probably will take Mathematics PS. Many of them will want a calculus course. Assuming adequate preparation for Mathematics 0, a reasonable sequence for such students is

$$PS \longrightarrow 0 \longrightarrow B$$

to which Mathematics FM may be added in the third or fourth semester.

Although Mathematics L is thought of as following

*See page 21, Pace and manner of presentation.

half of Mathematics C, it is possible for a bright student to take it after Mathematics B. The resulting sequence

$$0 \longrightarrow B \longrightarrow L$$

would be of great value as a mathematically stronger alternative to the more standard

$$0 \longrightarrow B \longrightarrow FM,$$

or $$0 \longrightarrow FM \longrightarrow B.$$

(iii) If the primary interest of a student is the inclusion of some mathematics as a part of his general education, then any of the courses PS, A, O, FM or the sequences already mentioned will, depending on his previous preparation, serve the purpose effectively.

CHAPTER II
THE BASIC UNIVERSITY PARALLEL COURSES

In this chapter we describe in some detail the courses we propose for the basic offerings and discuss reasons for the choices that have been made.

§1. Calculus Preparatory, Mathematics 0 and Mathematics A.

CUPM has taken the position that pre-calculus mathematics properly belongs in the high school (see GCMC, p. 9) but that many colleges may need to continue teaching courses at this level. In view of this need, a course, Mathematics 0, is described in GCMC. The same course is proposed for the two-year college, and is described below in somewhat expanded form.

In addition, recognizing the presence in two-year colleges of many students whose high school preparation is in need of reinforcement, the Panel suggests another course, Mathematics A. The subject matter of Mathematics A includes that of Mathematics 0, but the intended pace is much slower so that reviews of topics from arithmetic, algebra and geometry can be introduced and pursued at appropriate stages of the Mathematics 0 outline.

It is hoped that each two-year college will be able to adapt the basic idea of this course to its own needs, perhaps offering both Mathematics 0 for its well prepared students and a version of Mathematics A to meet the needs of the rest of its students who hope to complete some calculus. The courses, although designed for students who plan to take calculus, should carry credit for general education requirements. Most schools, however, have special general education courses, and would advise their use in preference to Mathematics 0 or A for the purpose.

MATHEMATICS 0, ELEMENTARY FUNCTIONS AND COORDINATE GEOMETRY

Discussion: The prerequisites for Introductory Calculus, Mathematics B (GCMC Mathematics 1) include the following two components, (a) and (b):

(a) Three years of secondary school mathematics. The usual beginning courses in algebra (perhaps begun in eighth grade) and geometry account for two of these years. The remaining year should include: quadratic equations; systems of linear and quadratic equations and inequalities; algebra of complex numbers; exponents and logarithms; the rudiments of numerical trigonometry; the rudiments of plane analytic geometry, including locus problems, polar coordinates and geometry of complex numbers; and arithmetic and geometric sequences.

(b) A course such as the following (or Mathematics A below):

COURSE OUTLINE FOR MATHEMATICS O

Elements of coordinate geometry. Plane Cartesian coordinate systems, projections, Euclidean distance; graphs; lines; slopes, parallel and perpendicular lines, vertical lines; circles.

Definition of function and algebra of functions. Various ways of describing functions; examples from previous mathematics and from outside mathematics; a general definition of function; the operations of addition, multiplication, composition and inverse; graphs of functions; constant functions, linear functions and the absolute value function.

Polynomial and rational functions. Definitions; graphs of quadratic and other polynomial functions; conics; locating zeros of polynomial functions; rational and irrational zeros; number of zeros; remainder and factor theorems; complex roots; rational functions and their graphs, including intuitive discussion of asymptotes; symmetry (even and odd functions), intercepts, other simple aids to curve sketching.

Exponential functions. Review of integral and rational exponents; arbitrary exponential function x^y--discuss need for definition in general case and indicate how it can be achieved in terms of approximations (intuitive idea of limit); graphs; exponential growth; applications.

Logarithmic functions. Logarithm as inverse of exponential; special bases of logarithms; graphs; applications.

Trigonometric functions. Review of numerical trigonometry and trigonometric functions of angles as ratios; trigonometric functions defined on the unit circle; mappings of the real line onto the unit circle (degrees, radians); trigonometric functions defined on the real line by composition of the foregoing two functions, graphs; in particular, sine and cosine combinations with various amplitudes and frequencies; periodicity; periodic motion; fundamental identities; polar coordinates and polar form of complex numbers; inverse trigonometric functions; graphs; trigonometric equations.

Functions of two variables. Introduction to three-dimensional rectangular coordinate systems; sketching graph of $z = f(x,y)$ by contour lines and plane slices in a few simple cases.

NOTE: The proposed first course in calculus (see p. 23) treats topics in analytic geometry only incidentally, as the calculus throws new light on the subject. Thus, rather than an integrated analytic geometry and calculus, it is more in the nature of a straight calculus course. It assumes as a prerequisite sufficient command of coordinate geometry for the study of single variable calculus and some familiarity with the elementary functions and general concept of function as outlined above.

MATHEMATICS A, ELEMENTARY FUNCTIONS AND COORDINATE GEOMETRY
(with Algebra and Trigonometry)

Discussion: This special course is designed for those students who, because of a weak mathematics background, are not prepared to begin an intense calculus preparatory course such as Mathematics 0. Many two-year colleges are currently meeting the needs of these students with courses in intermediate algebra, trigonometry, college algebra and analytic geometry. It is felt that the contributions of all such courses to the overall two-year college offerings would be achieved better through the medium of this single course, adapted to local conditions.

A two-year college might choose to offer only Mathematics A or only Mathematics 0, or both, depending on the needs of the student population.

When teaching courses of high school level in a two-year college, it is customary to repeat the material in essentially the same form as it was presented in high school. For students who were not successful in high school, this approach is often no more fruitful the second time than the first. The virtue of Mathematics A is its fresh approach to old topics. Instead of requiring courses that repeat early high school mathematics and are then followed by a course on elementary functions, it is suggested that a repetition of the high school material be interwoven with the topics of the latter course.

The notion of function is given a central and unifying role in Mathematics A. Functions and their graphs serve as a peg on which to hang the review of elementary material. They also provide a new perspective, as well as a method of illustrating the old material. In this way, it is thought that the necessary review can be presented in a sufficiently fresh and interesting manner to overcome much of the resistance that students may carry with them from high school. (Moreover, this new approach should make Mathematics A more pleasant for the faculty to teach.) The development of a sound understanding of the function concept provides a solid cornerstone on which to build additional mathematical concepts in later courses.

This course should serve at least the following three purposes:

(i) To prepare the student for calculus and other advanced mathematics by including the material of Mathematics 0. Rather than being concerned with precisely which topics should be included, however, the emphasis should be on developing the ability to understand and use mathematical methods at least at the level of Mathematics B.

(ii) To review and remedy deficiencies in arithmetic, algebra, geometry and basic logic:

(a) by means of assignments and class drill associated with Mathematics 0 topics. (Every opportunity should be seized to expose and stamp out abuses of logic and notation, and such common atrocities as $1/2 + 1/3 = 1/5$, $1/2 + 1/3 = 2/5$, $(a + b)^2 = a^2 + b^2$, and $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$, familiar to all teachers.)

(b) to interject discussions of review topics as the occasion arises in Mathematics 0 topics (see illustrations, p. 18).

(iii) To develop mathematical literacy. By this we mean the ability to read and understand mathematical statements and the ability to translate into mathematical language (making proper use of logical connectives) statements and problems expressed in ordinary English. Continual practice should be given in solving "word problems" and in analyzing mathematical statements, with particular emphasis on developing the ability to understand and to use deductive reasoning.

Because of the limited time available, some material must be slighted. In this instance, classical synthetic plane geometry is not covered in the detail or to the extent that often is done in high school. However, this is compensated by the inclusion of enough analytic geometry to provide sufficient geometric preparation for calculus.. Care has been used in choosing the topics in Mathematics A so as to include those topics which the student needs to progress successfully to calculus or to calculus-related courses. It nevertheless seems likely that five hours a week will be needed for Mathematics A during the first semester and possibly also during the second semester. It should be a slow course that includes a great deal of problem solving, and also lengthy excursions as suggested by student interests and needs. Although Mathematics A serves to prepare students for calculus, it should also make students aware of the power and beauty of mathematics. Every opportunity should be exploited by the teacher to provide examples and applications appropriate to the age and maturity of the students. Extensive use of "word problems" can be very effective in developing the ability to think mathematically and to use mathematics. We have attempted to give some indications along these lines in the course outline. In our view it is far more important to stir and develop the interest of the students than to cover each suggested topic.

It is essential that the students be forced throughout the course to translate English sentences into mathematical ones and vice versa in order to improve their mathematical literacy.

Unfortunately, there exists at this time no single textbook completely appropriate for a course such as Mathematics A. However, by drawing upon several existing texts, the teacher should be able to find suitable textual material, illustrative examples, and exercises with which to build the course and achieve the remedial objectives.

Finally, we believe that a student who is not capable of handling Mathematics A upon entering a two-year college will in all likelihood be unable to pursue mathematical subjects in a four-year college with profit. Such a student should not be considered a transfer student in mathematical subject fields.

COURSE OUTLINE FOR MATHEMATICS A

Concept of function. Introduction as a rule associating to each element of a set a unique element of another set. (There shouldn't be very much worry about the definition of the word "set." Many examples should be given, with special attention devoted to functions that cannot be easily represented by formulas. For instance, the number of birds in a given locality or on a given tree as a function of the time or of the individual tree; the number of female students at the college as a function of the year; a function with domain the set $\{A, B, C\}$ and range the set $\{D, E\}$; etc.) Cartesian coordinate systems and graphs of functions. Review of relevant geometry, such as basic facts concerning perpendiculars and directed line segments. Review of negative numbers, decimals and the arithmetic of fractions, keyed to the problem of representing functions graphically. Decimals reviewed in this vein. Relative sizes of numbers illustrated on the x-axis. Motivation by means of examples of the definitions of sum, product, difference, and quotient of functions. Review of rational operations with algebraic expressions. Graphing of inequalities. Review of fractions and decimals, with special emphasis on comparison of sizes. Simple probability as a set function, i.e., a brief discussion of possibility sets, truth sets, and the assignment of measures to sets and their subsets. Conditional probability.

Polynomials of one variable. Linear functions of one variable and their graphs. Slope of a straight line. Various forms of the equation of a straight line. The graph of a linear equation as a straight line. Quadratic functions of one variable and their graphs. Completing the square. Use of completion of the square to solve some simple maximum and minumum problems. (This provides an opportunity to introduce some interesting "word problems.") Definition of polynomial functions of one variable. Sum and product of polynomials. Laws of exponents for integer powers. Division of polynomials, with the process first taught purely as an algorithm (stress should be given to the similar process for integers, whereby one integer is divided by another and a quotient and remainder obtained; the formula $m = qn + r$). Examples, pointing out the desirability of finding zeros of polynomial functions and ranges where the functions are positive or negative. Review of factoring: common factor, quadratics and the quadratic formula, difference of squares, sum and difference of cubes. The factor and remainder theorems. Theorems on rational roots. The need for complex numbers. Complex numbers and their algebra (introduced as a natural extension of the real number system). Geometric interpretation of addition of complex numbers, multiplication by real numbers (i.e., the vector operations) and of multiplication (see illustrations, p. 18).

Arithmetic and geometric sequences. Definitions and sum formulas. Approximation (noting error term) of $\frac{1}{1 - x}$ by $\sum_{k=o}^{n} x^k$ for $-1 < x < 1$ (both by using the sum formula and by repeating the division algorithm). Intuitive discussion of the meaning of limit. (Application can be made to the fractional representation of an infinite repeating decimal. The limit concept can be illustrated by using both repeating and non-repeating infinite decimals.)

Combinations and the binomial theorem. Finite sets, their unions and intersections. Combinations and permutations of members of a set. The binomial coefficients $\binom{n}{k}$ as the number of combinations (subsets) with k members chosen from a set with n members. Additional simple probability, using the formulas for numbers of

permutations and combinations. More applications and "word problems" using poker, dice, baseball lineups, etc. Review of the manipulation of integer exponents. Direct evaluation of $(x + y)^n$ for $n = 2, 3, 4$. The binomial theorem for natural number exponents. Proof using the formula for number of combinations.

Rational functions and polynomials of more than one variable. Rational functions and their graphs, with particular attention given to factoring the denominators and to the concept of asymptote. Use of asymptotes to strengthen further the intuitive notion of limit. Linear inequalities. Simultaneous linear equations in two or three unknowns. Applications to elementary linear programming. Polynomial functions of two variables. Quadratic polynomials of two variables. Definitions and resulting equations of conics, but only for the axes parallel to the coordinate axes (no discussion of rotation of axes). Simultaneous solution of a linear and a quadratic equation in two variables, of two quadratics, and geometric interpretation of the solutions. (The student should be made aware of the importance of functions of several variables; that the initial restriction to one variable is only for the purpose of handling the simpler problem first.)

Exponential functions. Review of the laws of exponents, with explanations of why, with $a > 0$, a^0 is defined to be 1, a^{-x} to be $1/a^x$, and $a^{p/q}$ to be $\sqrt[q]{a^p}$ or $(\sqrt[q]{a})^p$. Real numbers reviewed and presented as infinite decimals. Discussion of the need to give meaning to an arbitrary real power. Discussion of approximations to specific numbers such as $10^{\sqrt{2}}$ or $(1.2)^{\pi}$ and how this concept might be used along with the limit concept to define such exponentials. (This provides a good opportunity to contribute another step toward the understanding of the limit concept. If a computer is available, convergence can be made to seem plausible by actually computing approximations to a number such as $(1.2)^{\pi}$, using the first few places in the decimal expansion of π.) General exponential functions. Sketch of the theory of binomial series for negative and fractional exponents, with more discussion of the limit

concept. Use of these in approximations of roots and powers. Applications of exponential functions to selected topics such as growth, interest or electricity.

Logarithmic functions. Discussion of the general notion of the inverse of a function. Application to the definition of the logarithm to base a as the inverse of the exponential to base a. Review of the use of common logarithms in calculation. The slide rule (only a few lessons, see illustrations, p. 18). Some discussion of the idea of approximating exponentials and logarithms by polynomials but with the approximating polynomials given without proof. (If a computer is available use it to verify that the approximating polynomials give values of the exponential and logarithm agreeing with those in tables. If time and interests permit consider $(1 + \frac{x}{n})^n$ as n increases, to relate compound interest and exponentials and to motivate polynomial approximations of e^x.)

Trigonometric functions. Review of simplest geometric properties of circles and triangles, especially right triangles (very few formal geometric proofs). Definition of trigonometric functions as ratios (illustrate, but do not belabor, problems of triangle solving). Trigonometric functions defined on the unit circle as functions of a real variable. Graphs of trigonometric functions. Symmetry; even and odd functions. Review of ratios, fractions and decimals once again. The following trigonometric identities: the elementary identities $\sin x/\cos x = \tan x$, $\sin(x + \frac{\pi}{2}) = \cos x$, etc.; the Pythagorean identities $\cos^2 x + \sin^2 x = 1$, $\sec^2 x = 1 + \tan^2 x$, $\operatorname{cosec}^2 x = 1 + \operatorname{cotan}^2 x$; formulas for $\cos(x \pm y)$, $\sin(x \pm y)$, $\cos x \pm \cos y$, and $\sin x \pm \sin y$. Law of sines and law of cosines. Trigonometric (polar) representation of complex numbers. Roots of complex numbers. Graphs of sums and products of trigonometric and exponential functions. Periodicity. Applications to periodic motion and other periodic phenomena. Inverse trigonometric functions using the general notion of inverse function. Their graphs.

Functions of two variables. Introduction to three dimensional coordinate systems, functions of two variables, and graphs of functions of two variables for a few simple cases.

Some illustrations: 1. Elementary probability theory provides an excellent opportunity to develop both the ability to reason deductively and the ability to translate "word problems" into mathematical language. Examples can be chosen from horse racing, tossing of dice or coins, drawing colored balls from bags, etc. The following is one of an endless variety of problems that are useful for developing the ability to analyze a problem and to assign probability measures to sets:

A man tosses two coins and then informs you that the outcome includes at least one "head". Determine the probability that the outcome consisted of two heads, given that the rule determining the man's statement is the following (two separate cases are considered):

Case (a) He says, "There is at least one head," precisely when this is true, otherwise he says nothing.

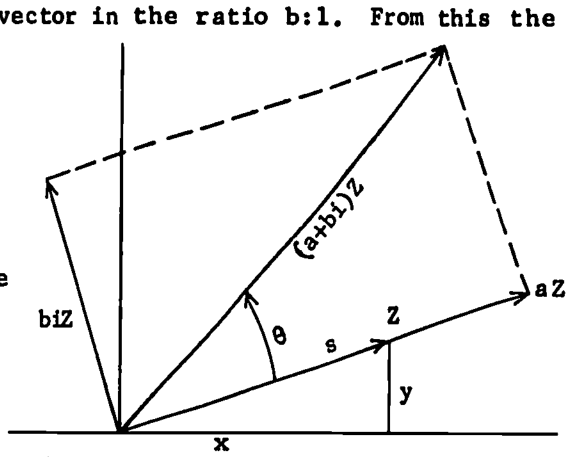
(The fact that he spoke means the outcome TT is eliminated from the four possibilities, leaving three equally likely cases and an answer of 1/3.)

Case (b) He says, "There is at least one head," when the outcome is HH. He says, "There is at least one tail," when the outcome is TT. In the remaining cases, he chooses which of these two statements to make on the basis of a coin toss.

(Under these rules, the outcome TT has probability zero in view of the statement actually made. The remaining elements of the sample space have probabilities: 1/2 for HH and 1/4 each of HT and TH. The answer is 1/2.)

2. The student will easily follow the interpretation of complex numbers as vectors and their addition as vector addition. He should also be taught to interpret this addition as a transformation of the plane. Thus $(a + bi) + (x + iy)$, where x and y are variables, defines a <u>translation</u> of the entire plane - in the fixed direction of $a + bi$. This remark is preliminary to the more difficult step (for the student) of interpreting $(a + bi)(x + iy)$ as an "operator" on the plane, a point of view which is of prime importance in many technologies.

This is broken down into steps: $a(x + iy)$ extends the vector $(x + iy)$ in the ratio a:1 [might be called an "extensor" or "amplifier" (one considers also the case that a is negative)]. Next $i[x + iy] = ix - y$ is a rotation (of the variable vector, and the entire plane) through $+90^{\circ}$ (i.e., counterclockwise), and $bi[x + iy]$ extends this new vector in the ratio b:1. From this the geometric meaning of $(a + bi)(x + iy)$ appears as a vector sum. The length of this sum vector is $s\sqrt{a^2 + b^2}$ where s is the length of $Z = (x + iy)$, its polar angle is as shown in the figure (where θ, the polar angle of $a + bi$, occurs as shown by reason of similar triangles).



Hence multiplication by $a + bi$ is an operator which rotates the plane through the angle θ and stretches all vectors in the ratio $\sqrt{a^2 + b^2}$ to 1. This gives the geometric interpretation of a product of complex numbers (multiply lengths and add angles) and later in the course it gives the addition formulas for sine and cosine (simultaneously). Of course, this geometric interpretation could also follow a discussion of trigonometry, as an application.

3. Word problems involving conjunctions and disjunctions are useful in sharpening logical thinking. For example:

> Thirty students are given grades 3, 2, or 1. If at least 10 got 3's and at most 5 got 1's, what can be said about the average score S for that class? [$3 \geqq S \geqq 65/30$]. What can be said about the number N of students who scored 2? If at most 10 got 3's and at least 5 got 1's what can be said about S? [$1 \leqq S \leqq 65/30$]. What can be said about the number N of students who scored 2?

Other, similar, problems can be introduced in a variety of contexts.

4. A brief discussion of the slide rule is of value in reinforcing the student's understanding of logarithms. Conversely, such an understanding can be the basis for learning the principles upon which the instrument is designed and give the student an adequate foundation for self-taught skill in its use.

Also, the student's interest in the slide rule can be exploited to great advantage in order to develop through practice his skill at estimating the magnitude of the result of an arithmetic computation (the problem of placing the decimal point).

§2. Calculus and Linear Algebra, Mathematics B, C and L.

This group of courses constitutes the proposed standard offerings for students expecting to major in mathematics, engineering and the physical sciences, but it is structured in such a way as to serve a variety of other needs. In particular, this is achieved by the course Mathematics B (similar to GCMC Mathematics 1). The main feature of this course is the inclusion of the main concepts of calculus (limit, derivative, integral and the fundamental theorem) in the setting of a single variable, dealing with the elementary functions studied in the earlier course and including computational techniques and applications of these ideas and methods. Thus the course provides a meaningful and usable study of the calculus for students who are unable to pursue the full sequence, as well as a desirable introduction for those who are.

Mathematics B can also be regarded as the calculus of elementary functions and, as such, forms a natural unit with Mathematics 0 or A: a thorough study of the elementary functions.

The remainder of the usual calculus topics, including series, functions of several variables and elementary differential equations, are described here as a single one year course, Mathematics C (GCMC Mathematics 2, 4).

Finally, we describe a course in linear algebra, Mathematics L, which is similar to GCMC Mathematics 3, only slightly less ambitious and employing a strong geometric flavor.

We consider it very important that the two-year college student who starts Mathematics C should complete it at the same school, rather than risk the probable discontinuity of study and loss of time attendant upon transfer before completion of this material. For this reason we have abandoned the feature of GCMC which proposes that linear algebra precede the study of functions of several variables and which

allows a somewhat deeper study of the latter. This feature constituted only one of the reasons for the inclusion of linear algebra in the lower division offerings. The remaining arguments apply as well to the two-year college situation, where the course can be taken simultaneously with the last half of Mathematics C in a two year sequence starting with Mathematics 0, and so will serve an important group of students.

Pace and manner of presentation.

Without intending to prescribe the exact format in which the proposed courses are to be offered, we nevertheless find it convenient to occasionally assume a system of semesters (roughly fourteen weeks of classes each) in order to state relative intensities of presentation in the familiar terms of 3, 4 and 5 hour* semester courses. In these terms we imagine the basic sequence of courses as embodying an increase of pace; the pace of the first courses taking realistic account of the wide range of student abilities and the last courses matching in pace the comparable courses at the transfer institutions.**

To illustrate, a distribution which fits our image of the average, multi-purpose community college is:

Mathematics A_1, A_2, 0 and B - 5 hours each

Mathematics C_1 - 4 hours

Mathematics C_2 and L - 3 hours each.

Such a progression of pace will, it is hoped, allow for enough additional drill and reinforcement of topics in the earlier courses, and enough attention to mathematical literacy, to provide for the student's growth in these areas to a degree of maturity at the end of the two-year college experience that puts him on equal standing with his fellow students at the best transfer school, where more severe screening may have taken place.

It is essential that this nurturing and growth process be achieved without weakening the content of the courses to the detriment of the better students. Indeed, for the better of the science-oriented students, it is important that these courses at the two-year college achieve as deep a penetration of the subject matter as

*These refer to the number of class meetings, not to the amount of credit granted, which we consider a local problem.

**To be sure, four-year colleges have students of varying abilities also, but there is more of a sink-or-swim philosophy than at many two-year colleges which regard the careful nurturing of students as a greater part of their function.

comparable university courses so as to prepare the student for more advanced courses in mathematics and related disciplines. A calculus course that fails to teach students to understand and analyze mathematics, that gives too few and too careless demonstrations of mathematical facts, will make later work very difficult for the student who attends a four-year college whose upper division curriculum presupposes more: students who have learned in high school to think mathematically may become disillusioned; students who have passed such a course may have false impressions of their mathematical knowledge; for all students, such courses mean the postponement of involvement with the true meaning of mathematics.

This, however, most emphatically does not mean that the course should strive for complete "rigor" as exemplified by, say, a detailed development of the real number system from axioms, preceded by formal discussions of logic and followed by careful epsilon-delta foundations for the calculus.*

It is possible to be intuitive and still be correct, simply by following intuitive motivations and plausibility arguments with precise statements and proofs where possible, and with honest omissions or postponements of these when necessary.

It is in this sense that Mathematics B is suggested to be an intuitive course (a certain amount of such omission is certainly indicated for the suggested amount of material to be covered). The gaps left in Mathematics B, the promises for more sound foundations, are intended to be partially filled in Mathematics C where, for example, questions of limit, continuity and integrability reappear (in the series and several variables settings).

Mathematics C, therefore, is intended to be a more sophisticated course, both in selection and treatment of subject matter, than Mathematics B. For example, we feel that the inclusion of a study of sequences and series in Mathematics C not only is important in itself and for its applications, but also is particularly effective for developing mathematical maturity. With the least upper bound property of the real numbers as the principal tool it is possible to give careful proofs of the convergence theorems for monotone bounded sequences and for series of non-negative numbers with bounded partial sums. Moreover, the study of sequences and series is a mathematical discipline that shows the need for precise definitions and demonstrates the dependence of mathematical understanding and mastery of formal techniques on basic theory. Also, the subject dramatizes the inadequacies of many of the student's prior intuitive ideas. It provides the student with perhaps his first major experience with a mathematical situation for which precise methods are indisputably

*In Chapter 5, section 5, we repeat some relevant statements on rigor that appear in GCMC.

needed! Indeed, it is necessary that some of the concepts introduced in Mathematics B, for example "limits," be further developed here.

MATHEMATICS B, INTRODUCTORY CALCULUS

Discussion: Mathematics B is designed to follow Mathematics O or Mathematics A. The purpose is to introduce the ideas of derivative and integral, motivated by geometrical and physical interpretations, and to study their interrelations through the fundamental theorem of calculus; to develop techniques of differentiation and integration; and to demonstrate the power and utility of the subject matter through frequent and varied applications.

Skills in manipulation need to be stressed in order that the student may learn to solve problems in the applications of calculus effectively, and to facilitate his study of later topics.

Continuing emphasis should be placed upon developing mathematical literacy: The lessons should stress correct interpretations of the written statements that set forth theorems, problem conditions and proofs.

COURSE OUTLINE FOR MATHEMATICS B

1. Slope of the secant line to a curve. Geometric discussion of the limiting position. Derivative as slope of tangent line. Average rates and instantaneous rates of change as intuitive motivation for derivative. Limit of difference quotient as special case of limit of function. Intuitive discussion of $\lim_{x \to a} f(x)$ including algebra of limits (describe the definition briefly using pictures and mention that the limit theorems hold for this definition). Definition of derivative based on intuitive notion of limit, algebraic properties - sum, product and quotient formulas with proofs.

2. Derivative of x^n, polynomials and rational functions. Point out continuity of polynomials (limit obtained by direct substitution). Examples of discontinuities, definition of continuity (illustrate with pictures). Derivatives and continuity properties of trigonometric functions reduced to showing $\lim_{h \to 0} \frac{\sin h}{h} = 1$. The existence of $f'(a)$ implies f is continuous at a.

3. Interpretations and applications: slope, velocity, rates of change, curve sketching, including increasing, decreasing, maxima and minima (confined to rational functions and simple combinations of

trigonometric functions). Need for more differentiation techniques.

4. Chain rule. Its importance to building techniques of differentiation. The obvious "proof"

$$\frac{df(g(x))}{dx} = \lim_{h \to 0} \frac{f(g(x+h))-f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} \cdot \frac{g(x+h)-g(x)}{h}$$

$$= f'(g(x)) \cdot g'(x),$$

breaks down because $g(x+h) - g(x)$ might be zero for arbitrarily small h. Difficulty is avoided either by a trick or by showing that in this case both sides are zero, yielding a proof (either do it or give a reference). Comments on the use of limit theorems and differentiability hypotheses in this argument.

General differentiation formulas for rational and trigonometric functions. Drill. Implicit differentiation. Derivatives of algebraic functions (general power formula) and inverse trigonometric functions.

5. More applications: word problems, related rates, curve sketching - need for more information on max, min, etc.

6. Intuitive discussion of Rolle's Theorem, mean value theorem. Proofs, assuming existence of maximum for continuous function on closed, bounded interval. Intermediate value theorem.

7. Higher derivatives and the differential as best linear approximation.

8. More applied problems. Curve sketching, rate problems and maximum and minimum problems with the more complete methods now available, acceleration, concavity.

9. Integration. (Motivate the integral as a limit of sums by observing how the latter arise naturally in several specific practical problems, e.g., area, work, moments, volume, mass, pressure. The integral is an abstract entity to embrace these and other quantities.) Discussion, with reliance on intuitive understanding of the limit involved. Antiderivative and fundamental theorem of calculus.

plausibility arguments for its validity and stress on its meaning for computation. Practice in finding antiderivatives, need for further techniques.

10. Exponential and logarithm. Definition of log x as $\int_1^x \frac{1}{t}\,dt$. Derivation of properties of log x from this formula. Exponential function as inverse of log x. Properties. Drill on complete list of differentiation formulas. Problems on applications involving these functions.

11. Applications of integration to finding areas, volumes of revolution, arc lengths, work done in emptying tanks.

12. If time permits: A more careful development of the definition of integral, discussion of properties of continuous functions and proof of the fundamental theorem.

MATHEMATICS C, MATHEMATICAL ANALYSIS

Discussion: Completing the traditional subject matter of calculus and an introduction to differential equations, this course serves also as the vehicle for an increasing sophistication, both in topics treated and the manner of their presentation.

Except for the obvious desirability of including the material on vectors prior to the beginning of Mathematics L (where, in any case, this material is briefly reviewed) no attempt has been made to divide the subject matter to fit quarters or semesters. There are two principal reasons for this. First, it seems important that students who intend to complete the entire calculus sequence (Mathematics B and C) should do it all at the same institution. Otherwise, wide variations in the order of topics can make transfer very difficult. Second, we leave to the particular college the problem of determining how many quarters or semesters and how many meetings per week are needed to present the material.

COURSE OUTLINE FOR MATHEMATICS C

Techniques of integration. Integration by parts. Integration by substitution, including use of the inverse trigonometric functions. Explanation of plausibility of general partial fraction expansion; use of partial fractions in integration. Emphasis on the use of tables of integrals. (Drill on techniques of integration provides useful practice in the manipulation of the elementary functions.

Stress should be placed on this rather than the development of ingenuity in integration techniques for its own sake.)

Limits. Limit of f(x) as x approaches a, where each deleted neighborhood of a contains a point of the domain of f. Right and left limits. Limits as x (or -x) increases without bound. Statement of basic theorems about limits with a proof of one or two of them and careful discussions of why all the theorems are reasonable and important. Continuity defined by the use of neighborhoods and then expressed in the language of limits. Continuity of sums, products, quotients, and composites of functions. Discussion of the intermediate value theorem; zeros of a function; Newton's method. Discussion of the maximum value theorem; proofs of mean value theorems; L'Hôpital's rule. Integral as a limit; the fundamental theorem via the mean value theorem; numerical integration and integrals.

Sequences and series. Definition of sequence as a function. Limits of sequences. Definition of series and sum of series. Relation to limit of sequence. Convergence of monotone bounded sequences and of series of non-negative numbers with bounded partial sums. Comparison, ratio, and integral tests for convergence of series. Conditional convergence. Power series and radii of convergence. Taylor's theorem with remainder. Taylor's series.

Vectors. Three-dimensional vectors: addition, multiplication by scalars, length, and inner (dot, or scalar) product. Parametric equations of lines and planes established by vector methods. Inner products used to find the distance between points and planes (or points and lines in a plane). Vector-valued functions and the position vector of a curve. Parametric equations of curves. Differentiation of vector-valued functions. Tangents to curves. Velocity as a tangent vector. Length of a curve. Polar coordinates. Curves in polar coordinates. Area in polar coordinates. Radial and transverse components of velocity for polar coordinates in the plane, with application to determining the length of a curve whose equation is expressed using polar coordinates. Area of surface of revolution.

If time permits: vector products, acceleration, curvature, and the angle between tangential and radial lines.

Differentiation. Partial derivatives. Differentials of functions of several variables. The chain rule for functions of several variables. Directional derivatives, gradients, and tangent planes. Theorem on change of order of partial differentiation. Implicit and inverse functions. Taylor's theorem and maxima and minima of functions of several variables.

Integration. Further use of simple integrals for computing such things as work, force due to fluid pressure, and volume by the shell and disk methods. Multiple integrals. Volume. Iterated integrals. Change of variables for polar, cylindrical, and spherical coordinates.

Differential equations.* Discussion of what an ordinary differential equation is; of ways in which ordinary differential equations arise; and of what it means for a function to be a solution of a differential equation, with verification of solutions in specific cases. Use of tangent fields for equations of type $y' = f(x,y)$ to build more understanding of the meaning of a differential equation and of its solution curves. Integration of first order equations with variables separable that arise in studying problems of growth and decay, fluid flow and diffusion, heat flow, etc. Integrating factors for first order linear equations, with more applications. Second order homogeneous equations with constant coefficients. Use of undetermined coefficients to solve the initial value problems of undamped and damped simple harmonic motion with forced vibrations. If time permits: more study of second order nonhomogeneous equations; series solutions.

*Many current calculus texts include a chapter which provides a brief introduction to differential equations, and closely related topics are often interwoven through other chapters (e.g., simple growth and decay problems in connection with logarithmic and exponential functions). Such treatments indicate the time allowance, if not necessarily the spirit, of the treatment which we have in mind.

MATHEMATICS L, LINEAR ALGEBRA

Discussion: There are perhaps three main reasons why a course in linear algebra should be offered by a two-year college that is able to do so. Such a course will contain the first introduction both to a kind of mathematical abstraction essential to the mathematical maturity of the student and, of course, also to serious mathematical ideas other than those of calculus. The subject is becoming more and more useful in the physical, biological, management and social sciences. Finally, it provides the proper setting for a deeper understanding of the basic notions of analytic geometry that the student sees in Mathematics O, A, B, and C.

In this program, the linear algebra course is designed to be a course parallel to the last part of Mathematics C. Of course vector ideas and methods should still be introduced as much as possible in the calculus sequence since they make possible a more meaningful development of that subject. Even though some basic vector ideas will be introduced in Mathematics C, the proposed linear algebra course can still be taken by capable students who have had only Mathematics B. To this end it begins with an introduction to geometric vectors, i.e., the course below has been designed in such a manner as to be accessible to a student who has not had vectors before.

In GCMC, linear algebra is recommended before the last term of calculus. This recommendation is gaining very widespread acceptance in the four-year colleges. However, the situation in two-year colleges is quite different and it is probably better there to allow the last part of calculus and the linear algebra to be taught in parallel, rather than to require one before the other. There is little question that most four-year colleges and universities would rather see the transfer student come in with the calculus completed than with some linear algebra at the price of having to complete the calculus. To finish calculus before transferring, the student may not have time to take linear algebra first, although he may well be able to take it simultaneously with the last part of calculus. Moreover, those transfer students who will major in the social or management sciences may take only one term of calculus but almost certainly will find linear algebra more useful than any other kind of mathematics. Here again, it is important that good students be able to take both the last part of calculus and the linear algebra.

The course outline below emphasizes a geometrical outlook and adheres closely to the guidelines for linear algebra suggested by the following quotation from GCMC.

> "When linear algebra is taught as early as the first term of the second year of college one should examine carefully the choice of topics. We need something intermediate between the simple matrix algebra which has been suggested

for high schools and the more sophisticated Finite-Dimensional Vector Spaces of Halmos. The pieces of linear algebra which have demonstrated survival at this curriculum level for many years are: Systems of linear equations and determinants from college algebra, uses of vectors in analytic geometry, and the calculus of inner products and vector cross products. More modern ideas call for the introduction of matrices as rectangular arrays with elementary row operations, Gaussian elimination, and matrix products; then abstract vector spaces, linear dependence, dimension, and linear transformations with matrices reappearing as their representations. Finally, of all subjects in undergraduate mathematics after elementary differential equations the one which has the widest usefulness in both science and mathematics is the circle of ideas in unitary geometry: orthogonality, orthogonal bases, orthogonal expansions, characteristic numbers and characteristic vectors."

COURSE OUTLINE FOR MATHEMATICS L

Geometrical vectors. Two and three dimensional vectors, sums, differences, scalar multiples, associative and commutative laws for these. Dot and cross products. Equations of lines in two and three dimensions, and planes by vector methods. Normals and orthogonality, direction cosines. Components of vectors and vector operations in terms of them. Mention correspondence between plane (space) vectors at the origin and R^2 (R^3).

Matrices and linear equations. Matrices, row and column vectors defined as rectangular arrays and n-tuples of real numbers. Sums of matrices and vectors, products of matrices. Examples and applications to elementary economics, biology, analytic geometry.* Linear equations written in matrix form. Gaussian elimination. Row echelon form of matrix. Existence and uniqueness theorems for solutions of homogeneous and nonhomogeneous systems of linear equations. Numerical examples for 2, 3, 4 variables.

*See, for example, Kemeny, Mirkil, Snell, Thompson, Finite Mathematical Structures, pages 205-223.

Vector spaces. Need for unifying concept to subsume n-tuples and geometric vectors of the previous paragraphs. Definition of abstract vector space over the reals. Examples of vector spaces from the above, including the null space of a matrix (solution space of a system of linear equations), function spaces, spaces of solutions of linear differential equations, space of polynomials of degree $\leq n$, of all polynomials. Linear combinations of vectors. Geometric applications. Linear dependence and independence. Theorem: If $m \geq n$ then m linear combinations of n vectora are dependent. Definition of dimension and proof of uniqueness (finite dimensional case only). Subspace. Geometric examples and function space examples. Dimension of subspaces; $\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$ with appropriate definitions and examples.

Linear transformation. Definition of linear transformation. Matrices associated with linear transformations with respect to different bases. Sums and products of linear transformations defined. (Show how this leads naturally to definitions of sums and products of matrices given in the second paragraph above.) Range and kernel of linear transformation. dim ker + dim range = dim of domain. Elementary matrices. Rank of linear transformation and matrix. Row rank = column rank. Linear equations re-examined from this point of view. Inverses of linear transformations and matrices. Calculation of inverse matrix by elimination. $PAQ = I_r$ if r = rank of A. Matrix of same linear transformation with respect to different bases. Similarity.

For the remainder of the course a choice may have to be made between the following two topics:

Determinants. The determinant as a function from matrices to the reals which is alternating and multilinear on the rows and columns, i.e., the determinant of a matrix is unchanged by adding a multiple of a row (column) to another row (column), the determinant of a diagonal matrix is the product of the diagonal entries, etc. (No attempt at proving existence in the general case.) Derivation of explicit formulas for 2×2 and 3×3 determinants. Calculation

of several explicit $n \times n$ determinants for $n > 3$, for example, Vandermonde 4×4 and 5×5 determinants. Expansion in terms of cofactors. If A, B are $n \times n$ matrices then determinant (AB) = determinant A · determinant B. (Last two topics with only indications of proof.) A very brief discussion of the formula AadjA = IdetA. Cramer's rule in determinantal formulation.

Inner product spaces. Definition. Orthogonal and orthonormal bases. Gram-Schmidt process. Distances. Orthogonal complements. Orthogonal expansions. Examples in function and polynomial spaces where inner product is defined via an integral. Applications to 3 dimensional analytic geometry.

§3. Probability and Statistics, Mathematics PS

The discussion and list of topics given here represents tentative recommendations arrived at jointly by some members of the CUPM Panels on Statistics and Two Year Colleges, in order to elaborate the latter Panel's recommendation that a course of this kind be included in the basic offerings.

Historically, a proliferation of introductory statistics courses, differing not only in content but also in emphasis and method of approach, has developed. Some of the factors responsible for this are the desire by different disciplines to have courses tailored to their needs, the varied backgrounds of persons teaching statistics, and differences in objectives for the basic statistics courses. The Panel on Statistics of CUPM is undertaking a full study of the difficult problem of developing models for such courses. The study will look to the future. It will investigate the role of computers, teaching devices and laboratories. It will consider special approaches, such as those stressing decision theory, non-parametric methods, Bayesian analysis, data analysis, and others. A major aspect of the study will be consultation with representatives of the various fields whose students are served by courses of this kind.

The study may well indicate that no single statistics course is best suited to fill the variety of existing and anticipated needs; the Panel may eventually recommend a choice from among several different types of statistics courses (hopefully providing at the same time the impetus for the appearance of any new texts they may require).

The projected recommendations will, of course, supersede the suggestions given here in that they will be more explicit and will be based on a more extensive study of the problem.

Discussion: An important element in the education of transfer students in many fields is a one term introductory course in statistics, including an adequate treatment of the needed topics from probability. Such a course is often required of students majoring in particular fields: to cite a few examples, business administration, psychology, sociology, forestry, industrial engineering. In addition, it serves as an excellent elective subject for other students.

A course of this kind can be intellectually significant although based on minimal mathematical preparation. In what follows a prerequisite of two years of high school algebra is assumed.

The main function of a first course in statistics which will be terminal for many students and, for others, will provide a basis for studying specialized methods within their majors, is to introduce them to variability and uncertainty and to some common applications of statistical methods, that is, methods of drawing inferences and making decisions from observed data. It is also desirable, as a second objective, for the students to learn many of the most common formulas, terms and methods.

Faculties of some departments served by this course stress the latter objective and, in addition, wish their students to be knowledgeable in certain specialized statistical techniques. We suggest that it is preferable for the students to learn basic concepts in the first course. (If a specialized technique is needed, it can be taught effectively in the subject matter course in the context in which it is used, at a small cost in time.)

It is quite plausible that an imaginative presentation which develops only a minority of the most common topics could be very effective in providing the students with an intelligent and flexible approach to statistical problems and methods through the development of basic concepts. Therefore, we regard the second objective of the course as secondary in importance.

Since the main objective of the course is understanding of the basic statistical concepts, proofs and extensive manipulations of formulas should be employed sparingly. While statistics utilizes these its major focus is on inference from data. By the same token, the course should not dwell upon computational techniques. The amount of computation should be determined by how much it helps the student to understand the principles involved.

We urge the use of a variety of realistic problems and examples, to help motivate the student and to illustrate the approach to such problems by statistical methods. (The use of a laboratory in conjunction with the course offers attractive possibilites for these purposes.) Examples drawn from disciplines of common interest to the class members should be sought. Also, the generation of data within the class (e.g., heights, birth dates) can be very useful.

Despite the diversity of basic courses, there has been a common set of topics included in most of them. The list below corresponds in an approximate fashion to this common set. A number of topics have been omitted from the list. Some because we feel they would detract from more important topics: combinatorial aspects of probability, skewness, kurtosis, geometric mean, index numbers and time series, for example. (These need not be avoided if they serve as pedagogical devices in attaining the major objectives of the course.) Other topics have been omitted because their incorporation would make it difficult to cover enough of the topics on the list. These include: Bayes theorem, Bayesian analysis, experimental design, sequential analysis, decision theory, and data analysis (insofar as this refers to an as yet emerging body of knowledge devoted to techniques for detecting regularities from data). These are potentially excellent topics; some of the possibilities for innovation in the basic statistical course are either (1) to stress some of these at the expense of omitting other (perhaps many) topics included in the list, or (2) to make one of these topics the basic focus for the entire course and discuss other topics in the framework thus established.

Finally, we urge that this course be a sound intellectual experience for the students. It should not be just a compendium of terms and techniques.

LIST OF TOPICS FOR MATHEMATICS PS

Note: This is not a course outline. It is not intended that this list of topics be necessarily covered in the order presented or in its entirety; an extremely skillful presentation would be required for a great majority of them to be combined satisfactorily in one course.

1. Introduction: Data: collection, analysis, inference. Examples of some uses of statistics emphasizing data collection, summarization, inference and decisions from data.

2. Probability: sample space, probability of an event, mutually exclusive events, independent and dependent events, conditional probability. (Keep enumeration to combinatorial notation at most. Use tables.) Random variable, expected value, mean and variance of a random variable.

3. Discrete random variable: binomial, use of binomial tables, Poisson (optional).

4. Summarization of data: Grouping data. Central tendency: arithmetic mean, median, mode. Dispersion: range, standard devia-

tion (others optional). Graphical analysis: histogram, frequency polygon, cumulative frequency polygon, percentiles.

5. Continuous random variables and distributions. Normal distributions. Use of tables. Normal approximation to binomial.

6. Sampling theory: Random sampling, Central Limit Theorem, normal approximation to distribution of $\bar{x}$.

7. Point estimation: Estimation of mean of the normal and mean of the binomial.

8. Interval estimation: Confidence intervals on the mean of the binomial, confidence intervals on the mean of the normal using the t distribution. Large sample approximate confidence intervals.

9. Hypothesis testing. Null hypothesis, alternate hypothesis, Type I and Type II errors, power of the test. Test of hypothesis on the mean of the normal using the t test. Test on the mean of the binomial. Large sample approximate tests.

10. χ^2 tests: goodness of fit, contingency tables.

11. Comparison of two population means. Paired and unpaired cases using the t test (non-parametric or range tests may be substituted).

12. Non-parametric methods. Sign test, Mann-Whitney test, Wilcoxon test. Confidence interval on median.

13. Regression and correlation. Least squares, correlation coefficient. Estimation and tests.

14. One way analysis of variance. F test, simultaneous confidence intervals.

CHAPTER III
TEACHER TRAINING AT TWO YEAR COLLEGES

One of the major concerns of CUPM since its inception has been the proper mathematical training of prospective teachers of school mathematics. Although much progress has been made over the years on this serious problem, a great deal remains to be done, and the two-year colleges will play an increasingly important part in these efforts. For this reason we have included in this report a statement from the CUPM Panel on Teacher Training, together with some excerpts from their current recommendations.

§1. Statement from CUPM's Panel on Teacher Training.

The following reports have been published by the Panel on Teacher Training: RECOMMENDATIONS FOR THE TRAINING OF TEACHERS OF MATHEMATICS (1961, revised 1966)*, COURSE GUIDES FOR THE TRAINING OF ELEMENTARY SCHOOL TEACHERS (revised, 1968) and COURSE GUIDES FOR THE TRAINING OF JUNIOR HIGH AND HIGH SCHOOL TEACHERS (1961). These three reports have been well received as a basis for the training of teachers, and have had the endorsement of such groups as the National Association of State Directors of Teacher Education and Certification.

At the time of the original recommendations the reform of mathematics in the schools was just beginning and it was not possible to see in all details the directions that would be taken by the curriculum at various levels. There have been many unexpected changes in the nature of school mathematics everywhere from Kindergarten to the twelfth year. The Panel has begun a study of what changes in its recommendations are needed to account for these developments and to prepare the teachers for the curricula of the next decades. The revised recommendations are planned for 1970 publication. Preliminary investigations indicate that all levels of the school curriculum will include additional topics from analysis, computing, applications of mathematics, probability and statistics--changes which will need to be reflected in the preservice training of teachers.

The secondary teacher will need four full years of mathematics, an amount equal to that of the regular mathematics major and differing from it primarily in that some of the courses he takes will be

*This report defines four levels of school mathematics and describes appropriate training for teachers at each level:

Level I - Teachers of elementary school mathematics.
Level II - Teachers of the elements of algebra and geometry.
Level III - Teachers of high school mathematics.
Level IV - Teachers of advanced programs in high school, etc.

specially aimed at the mathematics of the secondary school. Changes that occur at this level will probably be those occurring in the general college curriculum, together with the additional work that will be required of the prospective teacher to meet the needs of the new school curriculum. For this reason, it is important that the prospective teacher begin calculus as soon as his background permits. It is this need that the two-year colleges should keep in mind as regards the training of secondary school teachers. The courses 0, A, B, C and L of this report meet the need.

The Panel has made extensive efforts to explain its program for elementary school teachers to the colleges and universities: conferences have been held affecting every state (CUPM Report 15 gives a summary of these conferences).

A subsequent survey by the Panel shows that the amount of mathematics required of prospective elementary school teachers had approximately doubled in the period 1961-1966. In spite of these gains, the full implementation of the recommendations has not yet been achieved. There are many reasons for this outside the control of the mathematics departments, primarily the tremendous demands on the time of the prospective elementary school teacher.

Pending the revision discussed above, the Panel on Teacher Training believes that the courses described in the revised course guides offer the best preparation it can now describe for the elementary school teacher.

Most prospective elementary teachers are not highly motivated toward scientific and mathematical studies and are apt to be less well prepared than many other students. But, if we are to offer to children in the elementary grades a good mathematics program, indeed, if we wish to have the current commercial elementary textbook series well taught, we must manage to persuade these students that mathematics is an important discipline which they can understand. It is imperative that they begin studying the recommended two years of mathematics early, before they have lost too much contact with their earlier training, and in time to strengthen their mathematical backgrounds for other courses in their programs.

The Panel has found that one of the difficulties in putting the recommended courses into effect has been a tendency to make them too formal and too abstract for these students. The recent revision of the Level I course guides is intended to avoid this difficulty and indicate better the intended spirit of these courses.

Earlier in this report, it was recommended that all two-year colleges offer the year course in the number system, Mathematics NS. It should be emphasized that, in the view of the Panel on Teacher Training, this is <u>minimal</u> training for the elementary school teacher;

if at all possible, the college should offer the other year's work in algebra and geometry described in the revised course guides. If a choice must be made between these two courses, the intuitive geometry is preferred on the grounds that most students' prior training will be weaker in geometry than in algebra. By way of clarification of this observation, we quote here the objectives of the geometry course as set forth in the 1968 course guides:

A. To develop in the student an understanding of those fundamental geometrical ideas necessary for all who have occasion to work with mathematics. These include such concepts as congruence, measurement, parallelism, and similarity. To achieve sufficient emphasis on these, parts of other elegant and useful facets of geometry--such as concurrence of medians, altitudes, angle bisectors of triangles, and properties of angles inscribed in circles--are omitted from the main course, and left for inclusion in optional sections or in developmental problem sequences.

B. To develop geometric intuition and insight. This insight can best be obtained by doing. Of the several mathematics courses proposed in the Level I curriculum, geometry perhaps lends itself best to exercise in problem solving and to the development of techniques of problem solving; full advantage of this should be taken. We say again that the proof of theorems comes under the head of problem solving. Additionally, the student should be allowed to participate in the formulation of axioms which are based on intuitive plausibility arguments.

C. To develop an appreciation of a deductive system. In the outline, no single body of axioms is developed for all of Euclidean geometry. However, it is proposed that several deductive subsystems of Euclidean geometry, such as incidence geometry, should be formulated and developed. Traditionally, geometry has served as the first example of a rich deductive axiomatic structure. We should not like to see the prospective teacher have no contact with this aspect of mathematics.

D. To develop the ability to apply geometric ideas and to identify some practical applications of geometry.

E. To show that certain geometric systems can, with varying degrees of accuracy, describe the properties of the physical world.

§2. Recommendations of the Panel on Teacher Training.

RECOMMENDATIONS FOR LEVEL I
(Teachers of elementary school mathematics)

As a prerequisite for the college training of elementary school teachers, we recommend at least two years of college preparatory mathematics, consisting of a year of algebra and a year of geometry, or the same material in integrated courses. It must also be assumed that these teachers are competent in the basic techniques of arithmetic. The exact length of the training program will depend on the strength of their preparation. For their college training, we recommend the equivalent of the following courses:

(A) A two-course sequence devoted to the structure of the real number system and its subsystems.*

(B) A course devoted to the basic concepts of algebra.

(C) A course in informal geometry.

The material in these courses might, in a sense, duplicate material studied in high school by the prospective teacher, but we urge that this material be covered again, this time from a more sophisticated college-level point of view.

Whether the material suggested in (A) above can be covered in one or two courses will clearly depend upon the previous preparation of the student.

We strongly recommend that at least 20 percent of the Level I teachers in each school have stronger preparation in mathematics, comparable to Level II preparation but not necessarily including calculus. Such teachers would clearly strengthen the elementary program by their very presence within the school faculty. This additional preparation is certainly required for elementary teachers who are called upon to teach an introduction to algebra or geometry.

RECOMMENDATIONS FOR LEVEL II
(Teachers of the elements of algebra and geometry)

Prospective teachers should enter this program ready for a mathematics course at the level of a beginning course in analytic geometry and calculus (requiring a minimum of three years in college preparatory mathematics). It is recognized that many students will need to correct high school deficiencies in college. However, such courses as trigonometry and college algebra should not count toward the fulfillment of minimum requirements at the college level. Their college mathematics training should then include:

(A) Three courses in elementary analysis (including or presupposing the fundamentals of analytic geometry). This introduction

*This is the course which we have called Mathematics NS in the present report. See §3, p. 40 for the course outline.

to analysis should stress basic concepts. However, prospective teachers should be qualified to take more advanced mathematics courses requiring a year of calculus, and hence calculus courses especially designed for teachers are normally not desirable.

(B) Four other courses: a course in abstract algebra, a course in geometry, a course in probability from a set-theoretical point of view, and one elective. One of these courses should contain an introduction to the language of logic and sets. The Panel strongly recommends that a course in applied mathematics or statistics be included.

RECOMMENDATIONS FOR LEVEL III
(Teachers of high school mathematics)

Prospective teachers of mathematics beyond the elements of algebra and geometry should complete a major in mathematics and a minor in some field in which a substantial amount of mathematics is used. This latter should be selected from areas in the physical sciences, biological sciences, and from the social studies, but the minor should in each case be pursued to the extent that the student will have encountered substantial applications of mathematics.

The major in mathematics should include, in addition to the work listed under Level II, at least an additional course in each of algebra, geometry, and probability-statistics, and one more elective.

Thus, the minimum requirements for high school mathematics teachers should consist of the following:

(A) Three courses in analysis. These should be at least at the level of courses 1, 2, 4 of the CUPM report A GENERAL CURRICULUM IN MATHEMATICS FOR COLLEGES (1965).*

(B) Two courses in abstract algebra. The courses should include linear algebra as well as the study of groups, rings and fields. The courses 3 and 6 of the GCMC report are suitable.

(C) Two courses in geometry beyond analytic geometry. These courses should be directed at a higher understanding of the geometry of the school curriculum.

(D) Two courses in probability and statistics. These should be at least at the level of 2P and 7 of the GCMC report, and should be based on calculus.

*Mathematics B and C of this report.

(E) In view of the introduction of computing courses in the secondary school, a course in computer science is highly recommended. For example, the course described on p. 27 of the GCMC report and in the CUPM report RECOMMENDATIONS ON THE UNDERGRADUATE MATHEMATICS PROGRAM FOR WORK IN COMPUTING (1964), or the course "Introduction to Computer Science" described in the revised RECOMMENDATIONS ON THE UNDERGRADUATE MATHEMATICS PROGRAM FOR ENGINEERS AND PHYSICISTS (1967).

(F) Two upper-class elective courses. A course in the applications of mathematics is particularly desirable. Other courses suggested are, introduction to real variables, number theory, topology, or history of mathematics. Particular attention should be given here to laying groundwork for later graduate study.

One of these courses should contain an introduction to the language of logic and sets, which can be used in a variety of courses.

§3. The Structure of the Number System, Mathematics NS.

The following course description is taken from the CUPM report COURSE GUIDES FOR THE TRAINING OF ELEMENTARY SCHOOL TEACHERS (Revised, 1968).

COURSE OUTLINE FOR MATHEMATICS NS

1. INTRODUCTION.

This course is primarily for students with one year of study in each of high school algebra and high school geometry. Students with such preparation should be able to take the course with profit and without major difficulty. Students who have not had the equivalent amount of prior study may need to be supplied some background material. A person with a good command of the concepts, proofs, and techniques presented in the course should have the understanding of arithmetic necessary to teach in the elementary school.

A two-semester course taught in accord with this guide may at first appear to move fairly slowly. But it is intended that the course be thoroughly mastered; a superficial acquaintance with the concepts and techniques presented here will not be sufficient. A large part of the mathematics of the elementary school is arithmetic; consequently, the preparation must assure that a teacher have understanding and skill in arithmetic and confidence in his knowledge of the basic concepts of the subject.

The natural numbers should be introduced in relation to sets of real objects. The real numbers are introduced by successive extensions of the set of natural numbers. Each extension is carried out

for physical reasons (such as measurement) and for logical reasons (such as closure of the system under certain operations or general solvability of certain kinds of equations). Algorithms for performing operations can then be discovered and justified. It should be emphasized that there is not a unique algorithm for a given operation.

The use of simple equations and inequalities should be a prominent feature of the course, and the student should have extensive practice not merely in solving equations and inequalities, but also in formulating them to solve word problems.

It should be understood that this course guide is merely a brief sketch of the content of the topics that should be covered. In the case of Sections 2 and 3, it may be desirable to distribute the material throughout the course with a more structured discussion at the end, or in the middle, rather than the beginning of the course. Thus, after informally proving several theorems, a discussion of what constitutes a proof, and what does not, will make more sense to many prospective teachers than if this discussion is carried on without any specific proofs to consider. In a similar way, the concept of set can be used without a formal discussion of what a set is and without first introducing the formal notation. The notation can then be introduced later as a convenience.

The brevity of this sketch of topics creates an impression of logical austerity. It is, however, an essential part of the task of the teacher and textbook to avoid such austerity by filling in an intuitive background and furnishing illustrations at every opportunity. A bare sequence of definitions, theorems, and proofs is unacceptable for two reasons. In the first place, it would be pedagogically quite hopeless at the level, and for the audience, that we have in mind. In the second place, the prospective elementary school teacher needs to become aware of ways to bridge the gap between mathematical ideas as they appear in formal theories and the various intuitive forms in which these same ideas may be introduced to young children.

This point may be clarified by an example. In Section 7 the relevance of rational numbers to linear equations is discussed, and addition and multiplication are defined by certain formal rules. This treatment is logical. In a course, however, the introduction of rational numbers should be motivated through considerations of their historical development and their uses in the physical world, such as in measurement. To find the point $\frac{p}{q}$ on the number line, we should separate a unit segment into q separate parts, and lay p of these end to end, starting at 0. The rules of operation can be made plausible in these terms. The theory should come later, and should be checked against the intuitive model, to make sure that their properties agree.

We emphasize that the order of topics given here is not intended to be prescriptive. Nor is the list of topics meant to be definitive. Rather, this guide is meant to indicate the depth and extent of treatment envisaged by the Panel.

2. THE LANGUAGE AND NATURE OF DEDUCTIVE REASONING.

During the course, the students should acquire an understanding of the language and nature of deductive reasoning. It is not intended that the students should be introduced to formal logic, but rather that they should understand what it means to prove something. By the end of the course, they should be aware of the necessity of carefully worded definitions, the difference between axioms and theorems (and the need for axioms), the role in mathematics of such special words and phrases as: there is, for every, unique, one-to-one, if...then, if and only if, and, or, and not; they should know something about the nature of proof including the ideas of hypothesis, conclusion, implication, converse, and indirect proof.

The topics included above need not be the first topics studied in this course. Indeed, they need not be considered in a separate section. Alternatively, this material may be introduced piecemeal as the need arises, and as the opportunity presents itself.

3. ELEMENTS OF SET THEORY.

The notion of sets, with or without the notation, are so pervasive in mathematics that it seems desirable for school pupils and their teachers to have a clear understanding of these notions. It is not, however, desirable that every mathematical concept discussed in the course be reduced to set notation. This should not be a course in the foundations of mathematics, and demonstrating that certain portions of mathematics are as consistent as set theory is not one of our goals.

Using sets of objects to explain and motivate the various operations will be helpful. For example, arrays of dots or other objects will be useful in deriving an algorithm for multiplication. However, it is not necessary to introduce all of the notation of set theory. It is certainly undesirable to introduce it in one unmotivated section at the beginning of the course.

4. THE WHOLE NUMBERS.

The result of the process of counting is used to describe a property of a set, not to describe the physical objects which are the elements of the set. For example, if we look at a ram, a ewe, and a lamb, we may derive the following numbers: 3, the number of animals in the set of animals; 12, the number of hooves in the set of hooves; and 6, the number of eyes in the set of eyes. Thus, counting is a

process which assigns to every finite set a whole number. This idea should be explicitly the basis of the treatment of the whole numbers.

Then, each of the operations can be reduced to a counting process. For example, $17 + 53$ is the number of objects in a set which is the union of two disjoint sets, one with 17 elements and the other with 53 elements. This idea can be stated very informally at first, and then made more careful as the need for care becomes clear. Algorithms for addition are simply more efficient ways of doing the counting job. In a similar way, multiplication algorithms can be used as more efficient means of counting the objects in an array. If the base ten numeration system is understood, and grouping is used appropriately, the usual algorithm is easily derived. The commutative, associative and distributive properties can be verified for the operations on numbers by simply considering the corresponding operations with sets of physical objects. In preparation for future work with the real number line, a number line in which only the points corresponding to the non-negative integers are labeled should be discussed.

5. ADDITIONAL OPERATIONS AND RELATIONS.

The primary purposes of this section are to introduce subtraction and division and to develop an understanding of the rules for computation with positive integers. Subtraction and division may be introduced by use of the equations $a + x = b$, $ax = b$, $(a \neq 0)$, where a, b, and x represent whole numbers; or, they may be introduced as means of solving certain physical problems. Probably both methods should be used in a course for prospective elementary school teachers.

It is appropriate to begin this section with a discussion of order by defining $a<b$ if and only if there exists a positive whole number c such that $a + c = b$. It should be pointed out that points on the number line are labeled in such a way that $a<b$ if and only if the point corresponding to a is to the left of the point corresponding to b.

The usual order properties of transitivity, additivity, and multiplication by a positive whole number can now be motivated intuitively by appealing to the number line and then established by use of the definition. The relations $>$, $\leq$, $\geq$ can also be defined.

For $a<b$, the definition $b-a = c$ if and only if $a + c = b$ can now be formulated. The cancellation properties for addition and multiplication for positive whole numbers and the distributive property of multiplication over subtraction can be developed.

The definiton $b \div a = c$ $(a \neq 0)$ if and only if there exists a whole number c, such that $a \cdot c = b$ can be formulated and the elementary divisibility properties developed. A discussion of the division algorithm should follow. At this point it should be noted

that whole numbers are not closed under the operations of subtraction or division and that this inadequacy is provided for by the invention of other numbers to be studied in later sections.

6. NUMERATION SYSTEMS.

The fact that there is a difference between objects and their names (in particular, between numbers and numerals) should be mentioned, but not continually emphasized. An overemphasis on this distinction will be more confusing than helpful. Other points to be considered include the fact that there have been various numeration systems, most of them now obsolete, and the significance of base and place value in the decimal system. It is recommended that decimal and nondecimal systems be studied so that the teacher may acquire a better understanding of the basic structure and computational procedures in the decimal system. There should be no attempt for computational mastery in the nondecimal systems.

This section should begin with historical background including information about some of the ancient systems of notation, such as Babylonian, Egyptian, and Roman, and be followed by an introduction of place-value systems of numeration with bases other than ten and by an analysis of the procedures followed in adding, multiplying, subtracting, and dividing numbers whose symbols are in bases of ten and three, five, or twelve.

7. THE POSITIVE RATIONAL NUMBERS.

A logically complete treatment of the positive rational numbers is tedious. For example, to know that our definitions of addition makes sense, we need to check that if $\frac{a'}{b'} = \frac{a}{b}$ and $\frac{c'}{d'} = \frac{c}{d}$, then $\frac{(ad + bc)}{bd} = \frac{(a'd' + b'c')}{b'd'}$. The treatment in this course should, however, be complete enough that the student (1) will realize that these rules of arithmetic are matters of fact, dealing with properties of numbers and the real world, rather than matters of convention in manipulating symbols, and (2) will feel confident that the statements are true.

Motivation for the extension of the number system to the positive rationals should come both from a consideration of the use of numbers in measurement and from a desire to solve $ax = b$ where $a \neq 0$. The definitions for equality, addition, multiplication, and subtraction of rational numbers should be followed by the verification of the usual properties of rationals. In particular, it should be verified that the closure property under addition, multiplication, and division (except, of course, division by zero) holds. The commutative and associative properties under addition and multiplication and the distributive property of multiplication over addition or subtraction should also be developed. This section affords an excellent opportunity for improved competency in the elementary processes of

arithmetic, and improved understanding of algorithms for the basic operations.

The concept of order should now be extended to the positive rationals, and the identification of more points of the real line should take place. A discussion of ratio and proportion should be included in this section.

8. INTRODUCTION OF NEGATIVE RATIONAL NUMBERS.

The positive rationals were introduced as an extension of the whole numbers. Here the negative rationals are developed as a further extension of the number system. Just as $ax = b$ can be used to introduce the number $\frac{b}{a}$ so that $a\left(\frac{b}{a}\right) = b$, we can now introduce the number $-a$ so that $x + a = 0$ if x is replaced by $-a$. Again, there are also numerous physical examples that can be used to motivate this concept.

The number line should be used to develop intuitively the meaning of addition, subtraction, multiplication, and division of the numbers a and b where either one or both are negative rationals. This can be followed by the motivation of the definitions of the operations. This can be done by requiring that the commutative, associative, and distributive laws hold for negative rationals.

The zero product theorem ($pq = 0$ if and only if $p = 0$ or $q = 0$) should be developed. There should also be a presentation of the concept of absolute value and the trichotomy properties.

This course should provide abundant practice in computing with rational numbers, in using simple equations and inequalities to represent number relationships in word problems, and in finding solution sets of equations and inequalities.

The remarks made in the preceding section, about the pedagogic hazards of a complete logical treatment, also apply here.

9. ELEMENTARY NUMBER THEORY.

It is particularly important that the purpose and intended spirit of this section be clearly understood. The objective is not the mastery of an extensive body of knowledge, but it is to provide experience with (1) the discovery of some mathematical relations and theorems, (2) the making of simple proofs, and (3) the learning of the existence of famous and unsolved problems. Furthermore, the factoring ideas encountered here are beginning ideas which are expanded and generalized in the algebra course of this series.

This section should begin with a review and extension of the elementary divisibility properties and of the division algorithm.

A definition of the greatest common divisor and the least common multiple is appropriate at this point. Definitions of unit, prime, and composite numbers should be framed in a general manner so as to allow for the classification, not only of the elements of the set of positive whole numbers, but of the elements of an arbitrary subset of the set of positive whole numbers. A discussion of the classification of the positive whole numbers as unit, prime, or composite numbers should follow. Appropriate examples of the classification of elements of subsets of the set of positive integers, such as the even positive integers or the positive integers congruent to one modulo three, should be included.

With this background, it is feasible to prove the Euclidean algorithm and to introduce linear Diophantine equations in two variables. A complete proof by mathematical induction of the fundamental theorem of arithmetic (unique factorization) cannot be made at this time but indications of the proof can be given by means of an informal exposition which makes use of the phrase "in a finite number of steps." Examples should be given of sets whose elements do not factor uniquely. The set of even positive integers and the set of positive integers congruent to one modulo three serve this purpose. A proof of the infinitude of primes should be included.

We suggest that this section end with a discussion of some of the unsolved problems of number theory, e.g., the existence of infinitely many twin primes, Goldbach's conjecture and Fermat's last theorem.

10. DECIMALS AND THE REAL NUMBERS.

The purposes of this section are to develop the decimal representation of the rational numbers, to discuss the associated computational techniques and to use the decimal notation to introduce the concept of real numbers. That is, we shall define a real number to be an infinite decimal.

After the introduction of decimal notation for rational numbers, we find it possible to show that the computational devices developed for integers extend to terminating decimal numerals. Students should then discover that the subset of the rationals whose elements can be represented as terminating decimals is closed under addition, subtraction, and multiplication, but not division. The associative and commutative properties under addition and multiplication and the distributive properties can now be established for this system.

Pedagogically, it is probably best to introduce approximations at this stage before continuing with a development of the real number system. The notions of "rounding off" and scientific notation should be introduced here. The introduction of nonterminating decimals should be tied to the notion of approximation. For example, whereas

1/3 = .333..., the terminating decimals .3, .33, .333 are better and better approximations to 1/3, and .333... locates a point on the number line. It should be pointed out that the decimal representation need not be unique. Finally, it should be established that a number is rational if, and only if, its decimal repeats or terminates.

The real numbers can now be introduced as all those which can be represented as decimal numbers whether or not they terminate or repeat. The fact that every number represented by a decimal numeral locates a point on the number line can now be illustrated.

The number $\sqrt{2}$ may be used to show that there exist nonrational, real numbers. By the method of successive approximations, $\sqrt{2}$ can be shown to exist. This should be followed by a proof of the irrationality of $\sqrt{2}$ and the geometric construction of $\sqrt{2}$ on the number line by means of the Pythagorean theorem. Nonrepeating decimals can also be exhibited to reinforce the notion that nonrational numbers exist.

Algorithms for operations on real numbers should be discussed in terms of the operations on approximations to the real numbers.

Finally, the course may be profitably ended with a <u>very informal</u> discussion of the field, order, completeness and density properties of the real numbers. It should be clear that except for the completeness property, these properties hold for the rational numbers.

CHAPTER IV
OPTIONAL ADDITIONAL OFFERINGS

Many two-year colleges offer courses for transfer students other than a basic set of courses such as those described in Chapters 2 and 3. With the growing trend for well qualified and highly motivated students to attend two-year colleges the demand for such additional courses will increase. The opportunity to teach such courses will certainly be attractive to faculty members and will offer them stimulating challenges. In this chapter we offer some suggestions for the kind of additional courses which we think deserve prime consideration, including sample course outlines and discussions of the circumstances under which the course is deemed desirable.

§1. FINITE MATHEMATICS, MATHEMATICS FM

This course is designed primarily for students interested in business, management, social and biological sciences. Its purpose is to introduce a variety of mathematical topics, showing how these are related to problems in the areas mentioned above. It would be a valuable component of the curriculum only if a substantial amount of time, at least 20% of the course, were devoted to applications of the material developed. Simply taking up mathematical topics of potential relevance is not enough; indeed unless it includes a substantial unit on applications it would be better not to offer the course.

In finite mathematics the student is exposed to mathematical topics of a different nature than he has previously studied; topics that apply to new types of problems, and topics that are perhaps more closely related to his special interests. It is expected that some students, even if a small proportion, will be attracted to further study of mathematics by this course, either because they find these new topics interesting in themselves, or because they recognize the need to learn more for further applications.

The course described, while not requiring all of the topics in Mathematics 0, can profitably build on the mathematical sophistication and maturity of that level of mathematical education in order to develop the ideas needed for the applications. A course certainly can be designed for students who have completed less, although it may then be necessary to have the course meet an extra hour per week in order to have enough time at the end of the course for the desired substantial unit on applications. A course assuming Mathematics PS could proceed somewhat further with applications, either in greater depth or in greater variety. However, we believe finite mathematics will serve a greater need and a larger audience if statistics is not a prerequisite.

The topics suggested are chosen because of their applicability, and include finite sets, probability, and linear algebra. The elements of logic may be, but need not be, treated as a separate topic. All that is needed is an ability to analyze and interpret compound statements formed by the usual connectives "and," "or," "not," "if and only if," "if ... then ...," and familiarity with what is meant by a logical argument. These ideas can be woven into the course. We prefer the latter procedure since it will help insure time for applications.

Although it is not indicated in the outline, an attractive way of beginning is to present one or two applied problems that will be solved by the end of the course (e.g., one of the several Markov chain examples). As the course progresses and partial solutions are obtainable, the student should be apprised of the progress being made toward the ultimate solution to be given at the end of the course.

COURSE OUTLINE FOR MATHEMATICS FM

Sets. Subsets and set inclusion. The algebra of sets - union, intersection, complementation, difference. The relation with logic of compound sentences constructed from the connectives "and," "or," "not," "if and only if," "if ... then ... ." Tree diagrams as a systematic way of analyzing a set of logical possibilities. The number of subsets of a set. Cardinality of finite sets, card$(A \cup B)$ = card$A$ + card$B$ - card$(A \cap B)$. Partitions, binomial and multinomial theorems and related counting problems.

Probability. The concept of a probability measure - the axioms, and the realization of these axioms for finite sets. (The intuitive basis for this model discussed and illustrated by several examples.) The law of large numbers. Equiprobable measure. Probablistic independence. Repeated occurrences of the same event and binomial distribution. Approximation of the binomial distribution by the normal distribution as a means of making predictions about the likelihood of outcomes of repeated independent trials. (If time permits, also discuss Poisson distribution as approximation of the binomial.) Introduction to Markov chains (particularly if the course is begun by a discussion of a problem involving Markov chains - otherwise this topic may be delayed).

Linear algebra. Vectors and matrices introduced as arrays of numbers. Addition of vectors and matrices. Products (motivated by

such examples as: given a vector of prices of certain commodities, and a matrix giving the quantity purchased on each of a set of days, find the total expense on each day). Solution of linear equations, including the cases where the number of equations and number of indeterminates differ, and including the cases of multiple solution and no solution. The definition of the inverse of a matrix. Its computation without using determinants. Applications to Markov chains - classification of chains, with the emphasis on absorbing chains and regular chains. Fixed vectors and eigenvectors. Examples such as: for regular chains, analyze a problem in social mobility, probability of change of job classification, change of residence, change in brand preference among choices for a particular article, probability of change of political party allegiance, genetic heredity; for absorbing chains, many of these same examples with a change in the probabilites - e.g., brand preferences, genetic heredity, as well as matching pennies, learning processes.

Applications. One of the largest classes of these involves Markov chains, including the long term development of proportions of dominant and recessive genes, learning models, models on the judgment of the lengths of lines, the spread of a rumor in a housing development, or an extension of any of the items mentioned above.

Another attractive example is linear programming, including: an introduction to convex sets; the fact that a linear function defined on a convex set assumes its maximum and minimum at a vertex; some simple examples that can easily be solved by hand, with indication of the magnitude of the task in case the convex set has many faces and vertices. If time permits: A discussion of means of solving such problems, including the simplex method.

Still another example is the theory of games. Here one can introduce the idea of a zero-sum two-person game, especially a two-by-two matrix game, the idea of pure and mixed strategies, pure strategy solutions both for strictly determined games and those that are not strictly determined. Mixed strategies and the min-max theorem (proved for two-by-two matrix games). Examples - one of the simplified poker games; competition between two businesses.

§2. DIFFERENTIAL EQUATIONS AND ADVANCED CALCULUS, MATHEMATICS DE AND DA

Many two-year colleges now offer courses in differential equations. This practice is expected to continue because of the large number of students specializing in fields for which some familiarity with differential equations is important. Some students who take such a course do not transfer to four-year colleges and some of those who do transfer plan to major in fields which do not require many upper-division courses in mathematics at the four-year college. A two-year college might well offer a course in differential equations as an elective for such students, with Mathematics C and L as prerequisites. However, such a course has limited value if it trains students only in formal techniques for special cases without developing understanding of the general nature and meaning of differential equations and their solutions.

We regard linear algebra as being a more basic and important tool than differential equations for most students. For students who will take only one of the courses we strongly recommend linear algebra. Since the concept of vector space is very important when discussing general linear equations, linear algebra should be a prerequisite for differential equations. If students do not have this preparation, the treatment of systems of differential equations will, of necessity, be quite minimal.

The content of a differential equations course should be planned to meet the needs and interests of the students. Therefore, we have outlined a course, Mathematics DE, that has considerable flexibility and can be varied to meet such needs and interests. Certain important topics have been listed first. Other topics are listed from which choices can be made, either as additional topics or as substitutions. For example, the study of existence and uniqueness theorems might be substituted for the study of systems of equations, or numerical and operator methods might be introduced throughout the course.

For some students, further study of calculus is more important than the study of differential equations. This is true for many students planning to take mathematics after transfer. For such students, we recommend a course which develops some concepts of advanced calculus, but also provides an extension of the study of differential equations begun in Mathematics C. This course, Mathematics DA, would follow Mathematics C and L. Recognizing varied student and faculty interests and time limitations, we describe on page 53 the core material and some additional topics from which choices should be made as time permits.

The two courses whose outlines follow are designed to develop good understanding of the mathematics involved and to improve the

mathematical maturity and ability of the students. These courses are suitable only for students who have gained from courses such as Mathematics C and L the ability to understand and to readily assimilate mathematics. In particular, it is assumed that the students have studied an introduction to differential equations such as is described in the outline of Mathematics C.

COURSE OUTLINE FOR MATHEMATICS DE

Review. First order equations with variables separable; integrating factors for first order equations. Second order linear homogeneous equations with constant coefficients. The method of undetermined coefficients, with applications to undamped and damped simple harmonic motion with forced vibrations.

Linear differential equations. Superposition of solutions. Homogeneous equations of n^{th} order with constant coefficients; existence and uniqueness theorems for initial-value problems. Linear dependence and independence; vector spaces of solutions. Relation between solutions of a nonhomogeneous equation and the corresponding homogeneous equation. Methods of undetermined coefficients and variation of parameters. Applications to initial-value problems.

Series solutions. Discussion of term-by-term differentiation and integration of a power series within the interval of convergence; proof that the interval of convergence does not change, with at least heuristic justification of the procedure. Use of series for solving some first order and other simple cases for which convergence can be verified easily and the procedure justified. Discussion of both the method of undetermined coefficients and the use of Taylor series for determining series solutions. General theory of series solutions about regular points. Discussion of the indicial equation and of the nature of series solutions about regular singular points, with most results stated and only heuristic justification given. Applications to classical equations such as those of Bessel and Legendre. Some simple nonlinear equations, such as $y' = x^2 + xy^2$.

Systems of equations. Equivalence of general systems to systems of first order equations. Vector representation of a system of first order equations. Use of matrices to solve systems with constant

coefficients. Description of the nature of the solution when the coefficient matrix can be reduced to diagonal form, e.g., when the eigenvalues are distinct or the matrix is symmetric; description and application of the Jordan canonical form. Discussion of applications to mechanical and electrical problems.

Additional or alternative topics chosen from among the following:

1. Numerical methods. Difference equations and interpolation. Runge-Kutta method. Elementary considerations of stability and error analysis.

2. General linear equations. Wronskians; linear dependence and independence; number of linearly independent solutions of an ordinary linear differential equation.

3. Existence and uniqueness theorems. Convergence of power series solutions. Existence and uniqueness proofs for first order equations using Picard's method. Generalization to systems of first order equations.

4. Operator methods. The operator D for linear equations with constant coefficients; factoring and inversion of operators; partial fraction techniques. The Laplace transform applied to linear differential equations with constant coefficients and to simultaneous linear, first order equations with constant coefficients.

5. Nonlinear differential equations. Special nonlinear equations which are reducible to linear equations; local stability; simple phase-plane geometry of trajectories. Self-sustained oscillations of a nonlinear system. Forced oscillations of a nonlinear system [e.g., $x'' + k^2(x-cx^3) = A \cos ct$] and the corresponding resonance phenomenon.

COURSE OUTLINE FOR MATHEMATICS DA
(Differential Equations and Advanced Calculus)

Topology of the real line. Brief description of the construction of the real numbers from the rational numbers. Open and closed sets. Least upper bound property shown to imply Bolzano-Weierstrass and Heine-Borel properties.

Continuity. Preservation of connectedness and compactness by continuous maps; maximum-value, mean-value, and intermediate-value theorems; continuity of inverses. Uniform continuity.

The above topics should be accompanied by numerous examples and by counterexamples showing the need for the hypotheses.

Linear differential equations. Extension of the methods developed for second-order linear equations in Mathematics C. Linear dependence and independence; number of linearly independent solutions of an ordinary linear differential equation; vector spaces of solutions. Relation between solutions of a nonhomogeneous equation and the corresponding homogeneous equation. Methods of undetermined coefficients and variation of parameters. Applications to initial-value problems.

Convergence. Review of tests for convergence of series and sequences of constant terms. Algebraic operations with series and power series. Uniform convergence of sequences and series. Term by term integration and differentiation of sequences and series. Existence and uniqueness proofs for first order differential equations using Picard's method.

Additional or alternative topics chosen from among the following:

1. Series solution of differential equations. Power series. Use of power series to obtain solutions of differential equations. General theory of series solution about regular points. Discussion of the indicial equation and of the nature of series solutions about regular singular points, with most results stated and only heuristic justification given. Applications to classical equations such as those of Bessel and Legendre.

2. Systems of differential equations. Equivalence of general systems to systems of first order equations. Vector representation of a system of first order equations. Use of matrices to solve systems with constant coefficients. Description of the nature of the solution when the coefficient matrix can be reduced to diagonal form, e.g., when the eigenvalues are distinct or the matrix is symmetric; description and application of the Jordan canonical form. Discussion of applications to mechanical and electrical problems.

3. Riemann integration. Area and integrals. Properties of definite integrals. Existence of integrals of continuous and monotone functions.

4. Transformations. Review of partial differentiation and of linear transformations and matrices. Jacobians. Inverse transformation and implicit function theorems. Change of variables in multiple integrals. Cylindrical and spherical coordinates.

§3. PROBABILITY THEORY, MATHEMATICS PR

The importance of this subject makes it a prime candidate for inclusion early in the program of many transfer students. The usual treatment of the theory of probability requires some background in calculus. Sufficient background for this course is Mathematics B. The course itself will provide additional calculus material as the need arises.

This course should lay stress on problems and, in particular, on problems which provide motivation and develop interest in the conceptual aspects of probability theory. Problems of this kind can be found, for example, in Mosteller's book, Fifty Challenging Problems in Probability With Solutions (Addison-Wesley, 1965).

The course outline below is an adaptation of the course Mathematics 2P from the GCMC. The outline is brief. It is intended that the topics be treated in depth.

COURSE OUTLINE FOR MATHEMATICS PR

(a) Probability as a mathematical system. Sample spaces, events as subsets, probability axioms, simple theorems. Intuitively interesting examples. Special cases: finite sample spaces, equiprobable measure. Binomial coefficients and counting techniques applied to probability problems. Conditional probability, independent events, Bayes' formula.

(b) Random variables and their distributions. Random variables (discrete and continuous), density and distribution functions, special distributions (binomial, hypergeometric, Poisson, uniform, exponential, normal, ...), mean and variance, Chebychev inequality, independent random variables.

(c) Limit theorems. Poisson and normal approximation to the binomial, Central Limit Theorem, law of large numbers, some applications.

§4. NUMERICAL ANALYSIS, MATHEMATICS NA

A two-year college should consider offering a course in numerical analysis provided it has

(a) a curriculum of courses including all the "Basic Offerings"

(b) someone on the staff qualified to teach the course

(c) access to computing facilities with reasonably short turn-around time - a time-sharing system is ideally suited to this purpose, as is a calculator-computer that students can use "hands on."

The first two provisions above are not likely to be challenged. To fulfill the second it may be possible for some institutions to find a part-time teacher from local industry who is interested in the mathematical issues raised. As for computing facilities, while numerical analysis can be studied effectively without a digital computer, the actual computational processes involved can be tedious, and the course would be much less likely to attract students. If the only accessible computing equipment involves long delays between starting a program and getting the answers, the result is frustration and a discontinuity of interest in the problem being solved.

This course has several purposes. It is intended to capitalize on the interest students have in computers in order to spur their interest in mathematics and to kindle in them a desire to study further topics in mathematics, for in it they should find that turning a problem over to a computer eliminates some mathematical problems while raising others. In addition, it provides an introduction to the important techniques of numerical solution to a variety of problems, and introduces ways in which digital computers can be helpful in problem solution.

It should be recognized that this is not a course in computer science, but rather it is a mathematics course. It contains material closely related to the curriculum of any computer science program, however, and hence is appropriate for those who wish to pursue a major in computer science at the four-year college. This course is similar to course B-4 of the recommendations of the Association for

Computing Machinery.*

The course begins with a discussion of some of the problems that will be treated. Most of these will be problems the student is already acquainted with in some form; definite integrals that cannot be expressed in closed form, differential equations that he has not learned how to solve, as well as large systems of linear equations with coefficients containing more than two significant figures, and nonlinear equations in a single variable which in theory he may know how to solve, but which in practice are quite impossible.

The process of obtaining numerical solutions to these problems will stimulate the study of additional topics in analysis, linear algebra, probability, and statistics. The student will become concerned with approximations, rapidity of convergence, and error analysis.

Selections of topics from the outline will depend on how much, if any, attention to numerical techniques appeared in the calculus course, and on the tastes and interests of the instructor. It should in any case include examples of interpolation, approximation, and solutions of systems of equations.

COURSE OUTLINE FOR MATHEMATICS NA

Prerequisite: At least one year of calculus including infinite series.

Introduction. Some typical numerical problems; the theoretical aspects of numerical analysis such as convergence criteria, error estimates, versus the algorithmic aspects such as efficiency of an algorithm, error control. (Most of these topics can be nicely introduced through examples, e.g., the solution of quadratic equations or the problem of summing an infinite series.)

Interpolation and quadrature. The linear and quadratic case of polynomial interpolation; basic quadrature formulas; numerical differentiation and its attendant error problems. If time permits, discussion of the general Lagrange and Newton interpolation formulas, the Aitken method, and Romberg integration.

*Report of the Association for Computing Machinery (ACM), Curriculum 68, Communications of the ACM, March 1968 (available at one dollar from ACM Headquarters, 211 East 43 Street, New York, N. Y. 10017). This report contains a complete curriculum in computer science.

Solution of nonlinear equations. Bisection method, successive approximations, including simple convergence proofs, Newton's method, method of false position. Application to polynomial and other equations and to special interesting cases such as square and cube roots and reciprocals.

Linear systems of equations. The basic elimination step, Gauss and Gauss-Jordan elimination. Round-off error, the ill-conditioning problem in the case n=2. If time permits: brief introduction to matrix algebra and the inversion of matrices.

Solution of ordinary differential equations. Series solutions and their limitation. Euler's method, modified Euler's method, simplified Runge-Kutta.

CHAPTER V
IMPLEMENTATION

This final chapter is devoted to some comments on various aspects of the problem of implementing the proposals of the earlier chapters. It is never the intent of CUPM that its curriculum recommendations be "swallowed whole," but rather that they be used as a focus for the discussion of curricular problems on a national or regional scale, and ultimately that they serve on the local level as a starting point for improvements tailored to local needs. It is in this sense that CUPM wants to see this program implemented. We point out below how it can be considered in small pieces; how it can be staffed economically; how problems of articulation with high schools, four-year colleges and other two-year programs can be approached; and how such factors as the computer and the existing choices of textbooks may affect implementation.

We aim the following remarks at a generalized audience in full recognition of the fact that the actual readers represent diverse schools, and each will need to interpret them for his own needs.

§1. Implementation by Stages.

CUPM does not, of course, recommend a total alteration of current practices, but rather an evolution in certain directions.

The curriculum seems to divide naturally into parts, reasonably independent from the viewpoint that changes in one part need not force major changes in the others. Thus the proposed structure can be approached in stages, beginning wherever a department recognizes the greatest need:

(1) The introduction of Mathematics 0 and (or) Mathematics A in place of current calculus preparatory courses.

(2) The revision of the calculus courses along the lines of Mathematics B and C.

Remarks:

(a) If current practice includes highly integrated analytic geometry-calculus courses, these two stages are not quite independent. Nevertheless, the problem of proceeding with one of these stages and making appropriate adjustments in the other courses doesn't seem to be a major one.

(b) If stage 2 has been carried out successfully, the course Mathematics B should serve the students of non-physical sciences who need an introduction to calculus. Thus, some need for special purpose courses, such as business calculus, might be eliminated.

3. Introduction of Mathematics L.

4. Introduction of Mathematics PS.

Remark: Although courses of this kind are offered in most schools, frequently in departments other than mathematics, a well designed course given in one department can help avoid duplication of effort within a school.

5. Introduction or extension of the offerings for elementary school teachers.

6. Selection of appropriate additional courses.

We believe that none of these changes would create unusual demands in terms of the teaching staff they require. Indeed, that the entire basic program calls for only slightly more than the equivalent of two full time staff members is indicated by the following sample patterns.

	First Semester			Second Semester	
entry ⟶	A_1	(5 hours)*	⟶	A_2	(5 hours)
entry ⟶	B	(5 hours)	⟶	C_1	(4 hours)
	C_2	(3 hours)	⟶	L	(3 hours)

This provides the full science sequence with two points of entry and uses approximately the equivalent of one full time instructor.

	First Semester			Second Semester	
entry ⟶	0	(5 hours)	⟶	B	(5 hours)
	C_1	(4 hours)	⟶	C_2	(3 hours)
	L	(3 hours)	entry ⟶	PS	(3 hours)

This offers the additional natural point of entry and Mathematics PS once each year, again using about one instructor. The addition of the course Mathematics NS (3 hours each semester) and perhaps the second year of teacher training courses completes the basic offerings.

This simplified model provides considerable flexibility, perhaps even more than is needed by many of the smaller schools where the need to offer each course each semester may not be present. On

*See pages 21 and 22.

the other hand, it doesn't take account of multiple section courses, of the possibilities for more economy through such devices as a summer school entry to the science sequence, or of the additional flexibility that can perhaps be achieved with the quarter system. Nevertheless, it seems clear that departments working out more detailed models according to their own needs will find this program quite economical of manpower as compared to alternative curricula.

§2. Articulation.

Articulation efforts have as their primary goal establishing a procedure of transfer that is as smooth and convenient as possible for the student, thereby enabling him to be cognizant of the required courses in his field and to plan carefully whatever program he is pursuing. Although articulation between two-year colleges and four-year colleges encompasses many areas such as staffing, curricula, administration, and personnel, the main obstacle to be surmounted is that of "communication." Detailed information must be exchanged between the two, and that information must then be disseminated to all concerned.

As indicated in "Guidelines for Improving Articulation Between Junior and Senior Colleges"[1] there are five main aspects of the transfer from the two-year college to the four-year college. These are: administration; evaluation of transfer courses; curriculum planning; advising, counselling, and other student personnel services; articulation programs. We have kept these in mind in formulating the present recommendations.

Many states, however, have large, geographically dispersed systems of two-year colleges and four-year colleges in which a slow down or lack of communication causes some of the items listed in the "Guidelines" to become problems. Florida and California, for example, are states having geographically dispersed systems and which have experienced or foreseen such problems. They have organized statewide committees to discuss, study and recommend solutions or guidelines for the solution of these problems. The printed documents concerning their mathematics programs and articulation efforts are available from their respective state boards of education.[2]

The Advisory Council on College Chemistry published a report in January, 1967 entitled "Problems in Two-Year College Chemistry,

1. Available from American Council on Education, 1785 Massachusetts Avenue, N. W., Washington, D. C. 20036.

2. Report of the California Liaison Committee, November 1967, Sacramento. A Report on Articulation in Mathematics (Florida), June 1966, Tallahassee.

Supplement Number 20A."* This report deals with curriculum and articulation problems in the area of chemistry that are in many ways identical to those in mathematics.

The counselling, guidance, and placement procedures employed by the two-year college serve to minimize the articulation difficulties both between the high school and two-year college and between the two and four-year college. Complete information about the individual two-year college's mathematics program should be in the hands of local high school counselling staffs. In this manner the high school student, regardless of his background, can be informed as to where he should start his mathematics program in college. For example, a graduating student who suddenly realizes that he needs calculus for his major field, and who is not prepared, should be informed that the local two-year college offers courses such as Mathematics A and Mathematics 0 to correct the deficiency.

In general, the mathematics department is urged to exercise its responsibility for coordinating its work with high schools, general counsellors and transfer institutions. Two particular goals are to improve placement procedures and to establish sound criteria for transfer of credit.

Articulation between institutions can cease to be a problem only when complete communication has been established.

§3. General Education and Technical-Occupational Programs.

A further study of these topics will be a major part of the continuing work of the Panel. However, because of their great importance in two-year colleges a brief commentary on these subjects is included here.

General Education. A desirable way for a non-science major to acquire some understanding and appreciation of mathematics may be to take a course or courses in general education mathematics. This is true if the course is carefully planned and taught by skilled and inspiring teachers. Above all else, the course should be so designed and taught that the students discuss and do mathematics rather than only hear about mathematics.

No definite outline for such a course is suggested at this time. Each teacher should judge the best type of course to suit his

*Available from the Advisory Council on College Chemistry, Suite 1124, 701 Welch Road, Palo Alto, California 94304. Of special note in this document are the articles by W. T. Mooney, El Camino College, Calif.; P. Calvin Maybury, University of South Florida; and Arthur W. Gay, St. Petersburg Junior College.

class. CUPM hopes to offer him more specific assistance in this task soon.

Whatever is done should be meaningful relative to the student's experience and studies, and taught in a new light relative to his past experiences with the study of mathematics. In many ways college students are becoming more sophisticated, and hence a new approach is needed to reinforce old mathematical experiences and to develop new ones. It is desirable that the course achieve the minimum levels of understanding, skills and reasoning needed by all students regardless of their goals or preparation, and that it provide some base for continuation in mathematics for those students who can and should continue. The course should emphasize basic mathematical concepts of recognized importance for an educated person and should be oriented, where possible, toward meaningful applications. The material should cut across the traditional segmentation of arithmetic, algebra and geometry. The solution of "word problems" should be emphasized throughout.

Mathematics 0, A, PS and FM as described above can be very effective general education courses for students having the necessary preparation.

Technical-Occupational. Even though the majority of students who enter two-year colleges do so with the intention of transferring, there still exists a large group who seek training and instruction that will prepare them for employment at the end of two years. In order to accommodate the latter group of students, along with the portion of the first group who change their objectives, most two-year colleges provide a wide array of technical, semi-professional and occupational programs.*

The need for mathematics in these career programs varies widely with the program. Some require no more than simple arithmetic while others demand a working knowledge of mathematics ranging from elementary algebra through the calculus and differential equations. We feel that the course Mathematics A with examples drawn from technical areas has great potential as a service course for many such programs. If a program requires a more specialized course, it should be taught by a mathematics instructor who is familiar with the curriculum's objectives. The content and sequence in which the topics are taught should be determined by the coordinated efforts of the mathematics and career curriculum faculty members. The "integrated" or "related" mathematics of vocational programs may be taught more effectively by the vocational instructor.

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*The 1964 Annual Report of the American Association of Junior Colleges lists examples of technical and semi-professional programs to be found in two-year colleges.

Whenever possible, instruction and practice in the use of the computer for problem solving should be given in engineering and science oriented two-year occupational programs and use of the computer in data processing should be taught in business and health oriented two-year occupational programs.

§4. The Computer in Two Year Colleges.

The two-year college must be prepared for, and articulate with, the changes taking place in both secondary schools and four-year colleges in the area of computing and in the use of computers in mathematics courses.

Computers are rapidly being introduced into secondary schools both physically and as a part of the curriculum. Courses of three types are already being taught.

1) The first type of course serves as an introduction to and empasizes "computer science" as a distinct discipline incorporating the many facets of modern computers. It is exemplified by SMSG's course Algorithms, Computing, and Mathematics.*

2) The second type of course emphasizes instruction in a computer language. It includes an exposition of the essentials of a programming language and many examples of programs for the solution of various categories of mathematical problems.

3) The third type of course emphasizes the integration of computing into the mathematics curriculum. It uses the computer as a part of the instructional process in selected subject matter areas within mathematics.

To indicate the trend further, we note that SMSG is preparing a new curriculum intended to go from seventh through twelfth grades which emphasizes the use of algorithmic ideas and computation. Flow charting, for example, is introduced in the seventh grade.

In colleges, computing is being introduced in freshman mathematics. A number of universities have already done this in calculus. A CUPM newsletter is being written which cites some current programs and gives sample materials. A forthcoming report of the National Academy of Sciences is expected to recommend that the computer be taught to all freshman mathematics students as a part of their regular instruction in mathematics.

*School Mathematics Study Group, Stanford University, Stanford, California 94305.

The reasons for this far reaching change appear to be pedagogical, substantive, and social.

Pedagogically, the computer is a mathematical instrument and a laboratory. Computing facilitates, extends, and enriches learning in mathematics; mathematics extends and facilitates and is essential to learning in computing. The mathematics class is the prime place in the two-year college for learning computing. Computing in the mathematics class can aid understanding, help to build concepts, and develop mathematical intuition.

Substantively, the student's education is broadened to include an introduction to computer science and numerical analysis.

Socially, the computer is becoming an essential part of many aspects of the student's life and work and is an excellent means of demonstrating the relevance of mathematics to the needs of society and the individual. In addition, training in the use of the computer is likely to provide the two-year college student with a skill which will open the doors to a variety of employment opportunities otherwise unavailable to him.

Every transfer student, with the possible exception of humanities majors, should at some time in his college career take a course in computer science such as the course B-1 of the ACM recommendations.* We make this recommendation both because of the direct importance of the computer in many fields of knowledge and because we feel the students should be aware of the capabilities and limitations of the computer.

The two-year college should be prepared to introduce computing into mathematics courses wherever relevant and appropriate. All of our recommended courses can make good use of flow charts, algorithms, and computer programmed assignments.

Support for these recommendations can be found in the studies that have been and are being made of the role of the computer in higher education. See, for example, the report of the President's Science Advisory Committee, "Computing in Higher Education," February 1967 (the Pierce report).

§5. Implementation of Individual Courses.

(a) Applications.

At all times in teaching these courses, every effort should be made to illustrate and motivate the material in them by showing how

*See footnote p. 57 for the reference.

it is used in a large variety of applications. Furthermore, the basic orientation should be towards the solution of concrete problems and at all costs the deadly dry "definition, theorem, proof" approach should be avoided. Of course, there should be carefully stated definitions, theorems, and some proofs, especially in Mathematics C and L as well as in the post-calculus courses, but, especially in the earlier courses, PS, O, A, B and FM, the problem-solving approach should predominate and new concepts should only be introduced after adequate motivation and examples.

One of the main aims of the teacher should be to get as active student participation in these courses as possible, by involving them in class discussions and a great deal of homework, which, of course, must be adequately criticized and discussed. One of the best ways to awaken and stimulate students' interest is constantly to demonstrate the many applications to a large variety of areas of knowledge that their course material has. After all, a great deal of the mathematics in this program originated in attempts at solving very definite real world problems.

Courses O, A, and FM offer many opportunities to include applications to elementary probability and combinatorics. In the latter course some mention of questions answered by linear programming techniques can also be made. In the calculus courses the usual applications should, of course, be done but, at all times, the teacher must be on the lookout for unusual applications as, for example, those given in the last part of S. Stein's book, Calculus in the First Three Dimensions (McGraw-Hill, 1967). The technical literature is full of applications of the calculus to subjects such as biology and the social sciences and problems such as traffic flow, pollution, etc. Every calculus teacher has an obligation to try to find out at least something about them. Mathematics L also has many applications in these areas and the same general remarks apply.

(b) Rigor.

The following quotation from GCMC reflects our views.

"Rigor. Even if it were possible to make a precise statement about the level of rigor we expect in this program, that would be undesirable, for the appropriate level of rigor varies from classroom to classroom and we should not attempt to dictate what it should be. Let us agree that it is the level of rigor in the student's understanding which counts and not only the rigor of the text or lecture presented to him. Our definite proposal is that the advancement of a student from the first calculus course (Mathematics B) to the second shall not depend upon his being able himself to make rigorous epsilon-delta proofs. The problem, as we see it, is to devise a presentation which, while not strictly logical at all stages, will nevertheless be intelligible, and which will convey the ideas of the calculus in forms which are intuitively valid, can later be made exact, and are made rigorous as the student advances.

We shall make no attempt to give a definition of what we mean by a valid first exposition. It seems better to give some illustrations.

(1) It was known to the ancients that the area of a circle is given by the formula

$$A = \pi r^2.$$

We realize today that this formula does not have a meaning until area has been defined; and Jordan measure, to which the ancients tacitly appealed, was not defined until the nineteenth century. Nevertheless we cannot deny that meaningful mathematics was based on the formula well before that time. Thus mathematical understanding is a matter of degree, and some kinds of informal understanding can be adequate for a long time.

(2) In a similar way, a student may get along, at least for a while, without the formal definition of a limit. But limits, and all other concepts of the calculus, should be taught as <u>concepts</u> in some form at every stage. For example, the fundamental theorem of the integral calculus involves two concepts: the "limit" of a sum and the antiderivative. Supposing that f is continuous and

$$\int_a^b f(x)\,dx$$

has been defined by approximating sums, the theorem states that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F' = f$. There is, to begin with, no obvious relation between the two sides of this equation, and only a proof can make it credible. One natural proof depends on proving, starting from the definition of an integral, that if

$$G(x) = \int_a^x f(t)dt$$

then $G' = F' = f$, whence G and F differ by a constant, which can only be $F(a)$. Thus a simple test to determine whether a student understands the fundamental theorem is to ask him to differentiate

$$G(x) = \int_o^x \sqrt{1+t^8}\,dt.$$

If he does not know how, he does not understand the theorem. It is dishonest to conceal the connection between the two concepts by conditioning the student to accept the formalism without his being aware that the concepts are there. On the other hand, to give the student only the concepts without making him fully aware of the formalism is to lose sight of the aspect of calculus that makes it such a powerful tool in applications as well as in pure mathematics.

(3) A "cookbook" would teach students to solve maximum problems by setting the derivative equal to zero and accepting the result, perhaps after "testing" to see that the second derivative is negative. A thoroughly rigorous book would demand careful proofs of the existence of a maximum of a continuous function, Rolle's theorem, and so on. What we suggest for the first calculus course is a clear statement of the problem of maximizing a function on its domain, a precise statement of such pertinent properties as the existence of the maximum, and the examples to indicate that the maximum, if it exists, is to be sought among endpoints, critical points, and points where the derivative does not exist.

(4) These remarks are meant as descriptions of the style in which the study of calculus should begin, not of the style in which it should end. One of the ultimate objectives is an exact mathematical understanding; and the question as we see it, is not whether we should achieve this, but rather when, and how, and after what preliminaries. This is a delicate question and we have no simple answer to offer. One possibility is a "spiral" technique, in which every concept is attacked several times. Or we might proceed informally up to a certain point, and then formalize. But in any case, we should start where the student really is, and proceed to where he should be. Mathematical sophistication should be introduced little by little."

(c) Textbooks.

Obviously the textbooks for all the courses will have to be chosen with a great deal of care. The teacher should make a continuing effort to keep abreast of new books as they appear. Moreover, while teaching any given course from a fixed book, the teacher should be consulting several other books for additional problems, extra illustrative material and different slants on presentation. Clearly, local conditions will often play a determining role in the choice of books, but at all times the instructional staff should be aware of what other books exist for the same subjects and make good use of these additional sources.

The Basic Library List issued by CUPM in 1965 and the forthcoming Basic Library List for Two Year Colleges should prove particularly helpful in making choices of texts and reference works for the courses described in this report. Moreover, the book review sections of the American Mathematical Monthly, the Mathematical Gazette, Mathematics Magazine, Mathematics Teacher, School Science and Mathematics and SIAM Review often will have helpful comments about new books and should be consulted regularly.